ECE 2025 Problem Set #9 Solution

9.1 a) use the sampling property of impulses

$$\delta(t-3.5) = \begin{cases} 1, & t = 3.5 \\ 0, & \text{else} \end{cases}$$

and L'Hôptial's rule

Using L'Hôpital's rule, we have

$$\lim_{t \to 3.5} \frac{\sin(13\pi(t-3.5))}{\sin(\pi(t-3.5))}$$

$$= \lim_{t \to 3.5} \frac{\cos(13\pi(t-3.5)) \cdot 13\pi}{\cos(\pi(t-3.5)) \cdot \pi} = 13$$

Hence

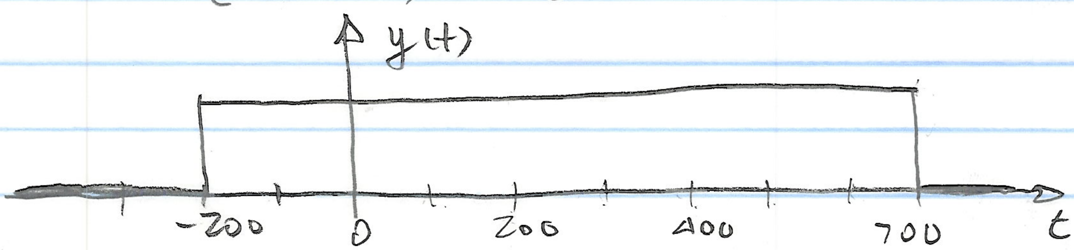
$$x(t) = \frac{\sin(13\pi(t-3.5))}{\sin(\pi(t-3.5))} \delta(t-3.5)$$

$$= 13\delta(t-3.5)$$

b) use property of impulse $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

$$y(t) = \int_{-200}^{700} 3.3\,\delta(\lambda - t)\,d\lambda$$

$$= \begin{cases} 3.3, & -200 \le t \le 700 \\ 0, & \text{else} \end{cases}$$



c) use chain rule of derivatives and sampling property of impulses.

Let

$$x(t) = \frac{d}{dt}\left\{0.7\cos\left(\tfrac{1}{2}\pi t\right)u(t-6)\right\}$$

$$= 0.7\cos\left(\tfrac{1}{2}\pi t\right)\frac{d}{dt}\{u(t-6)\}$$

$$+ \frac{d}{dt}\left\{0.7\cos\left(\tfrac{1}{2}\pi t\right)\right\}u(t-6)$$

$$= 0.7\cos\left(\tfrac{1}{2}\pi t\right)\delta(t-6)$$

$$- 0.35\pi\sin\left(\tfrac{1}{2}\pi t\right)u(t-6)$$

Then

$$z(t) = x(t) * \delta(t-0.5)$$

$$= x(t-0.5)$$

$$= 0.7\cos\left(\tfrac{1}{2}\pi(t-0.5)\right)\delta(t-0.5-6)$$

$$- 0.35\pi\sin\left(\tfrac{1}{2}\pi(t-0.5)\right)u(t-0.5-6)$$

$$= 0.7\cos\left(\tfrac{1}{2}\pi(6)\right)\delta(t-6.5)$$

$$- 0.35\pi\sin\left(\tfrac{1}{2}\pi(t-0.5)\right)u(t-6.5)$$

$$= -0.7\delta(t-6.5)$$
$$- 0.35\pi\sin\left(\tfrac{1}{2}\pi(t-0.5)\right)u(t-6.5)$$

d) use sampling property of impulses.

$$s(t) = \int_{-\infty}^{\infty} 9e^{-0.4\tau^2}\,\delta\left(\tau - \tfrac{1}{2}t\right)d\tau$$

$$= 9e^{-0.4\left(\frac{1}{2}t\right)^2}$$

$$= 9e^{-0.1t^2}$$

2a) use linearity and the property

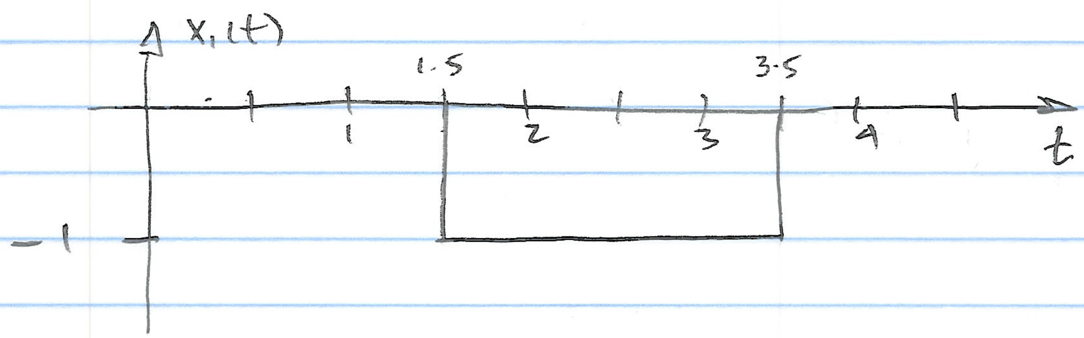
$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$x_1(t) = u(t-1/2) * [\delta(t-3) - \delta(t-1)]$$

$$= u(t-1/2) * \delta(t-3) - u(t-1/2) * \delta(t-1)$$

$$= u(t-3.5) - u(t-1.5)$$

$$= -u(t-1.5) + u(t-3.5)$$



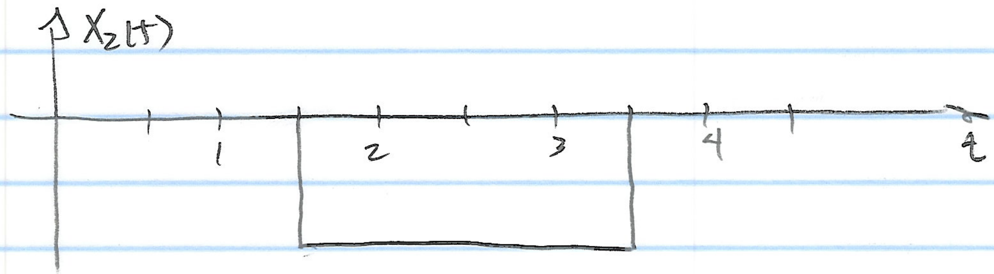
b) use communitive property of convolution and the property

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$x_2(t) = \delta(t-1/2) * [u(t-3) - u(t-1)]$$

$$= [u(t-3) - u(t-1)] * \delta(t-1/2)$$

$$= -u(t-1.5) + u(t-3.5)$$



c) convolution of a unit step and rectangular pulse

$$x_3(t) = u(t-1/2) * [u(t-3) - u(t-1)]$$

Note

$$= \int_{-\infty}^{\infty} [u(\tau-3) - u(\tau-1)]\, u(t-\tau-1/2)\, d\tau$$

$$= -\int_{1}^{3} u(t-\tau-1/2)\, d\tau$$

Now consider ranges of t

For $t < 1.5$, $\quad x_3(t) = 0$

For $1.5 \le t \le 3.5$

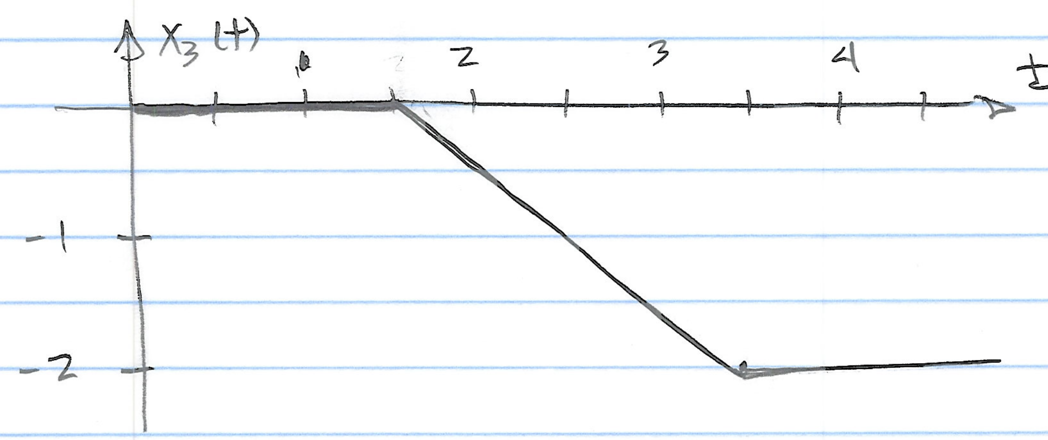
$$x_3(t) = -\int_1^{t-1/2} d\tau = -\tau\Big|_1^{t-1/2}$$

$$= -t + 1/2 + 1$$

$$= 1.5 - t$$

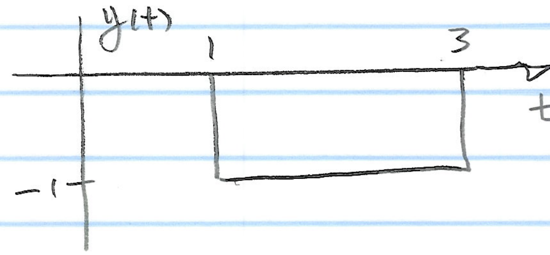
For $t > 3.5$

$$x_3(t) = -\int_1^{3} d\tau = -2$$



$$x_3(t) = \begin{cases} 0, & t < 1.5 \\ 1.5 - t, & 1.5 \le t \le 3.5 \\ -2 & t > 3.5 \end{cases}$$

d) convolution of two rectangular pulses.



let $y(t) = u(t-3) - u(t-1)$

Then $x_4(t) = y(t) * y(t)$

$$= \int_{-\infty}^{\infty} y(\tau)\, y(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \left[u(\tau-3) - u(\tau-1)\right] \times \left[u(t-\tau-3) - u(t-\tau-1)\right] d\tau$$

$$= \int_{1}^{3} \left[u(t-\tau-1) - u(t-\tau-3)\right] d\tau$$

Consider ranges of t

For $t \leq 2$, $x_4(t) = 0$

For $2 \leq t \leq 4$

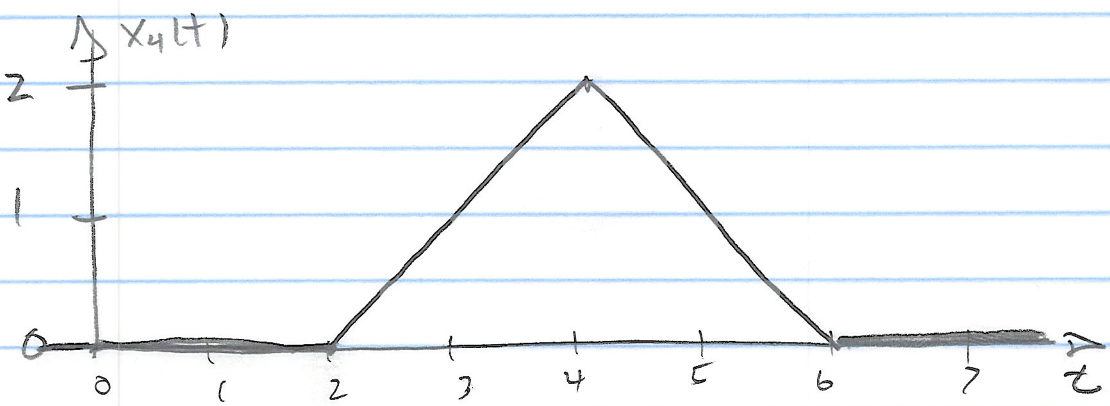
$$x_4(t) = \int_{1}^{t-1} d\tau = t-1-1 = t-2$$

For $4 \le t \le 6$

$$x_4(t) = \int_{t-3}^{3} d\tau = 3-(t-3) = 6-t$$

For $t \geq 6$ $\quad x_4(t) = 0$

$$x_4(t) = \begin{cases} 0, & t \le 2 \\ t-2, & 2 \le t \le 4 \\ 6-t, & 4 \le t \le 6 \\ 0, & t \geq 6 \end{cases}$$



9.3a) Use linearity and superposition, to write

$$x(t) * h(t) = x(t) * h_1(t) - x(t) * h_2(t)$$

where

$$h(t) = h_1(t) + h_2(t)$$

and
$$h_1(t) = \pi\,\delta(t-2.5)$$
$$h_2(t) = \frac{1}{2}\,u(t-3.5)$$

1. $x(t) * h_1(t)$

$$= x(t) * \pi\,\delta(t-2.5)$$
$$= \pi\, x(t-2.5)$$
$$= \frac{\pi}{3} e^{-3(t-3.5)} \left[u(t-2.5) - u(t-3)\right]$$

2. $x(t) * h_2(t)$

$$= \frac{1}{3} e^{-3(t-1)} \left[u(t) - u(t-1/2)\right] * \frac{1}{2} u(t-3.5)$$

$$= \frac{1}{6} \int_{-\infty}^{\infty} e^{-3(\tau-1)} \left[u(\tau) - u(\tau - 1/2)\right] \times u(t-\tau-3.5)\, d\tau$$

$$= \frac{1}{6} \int_{0}^{1/2} e^{-3(\tau-1)}\, u(t-\tau-3.5)\, d\tau$$

we now break into regions of t.

Case 1: $t < 3.5$, $x(t) * h_1(t) = 0$

Case 2: $3.5 < t < 4$

$$x(t) * h_2(t) = \frac{1}{6}\int_0^{t-3.5} e^{-3(\tau-1)}\,d\tau$$

$$= \frac{e^3}{6}\int_0^{t-3.5} e^{-3\tau}\,d\tau$$

$$= -\frac{e^3}{18} e^{-3\tau}\Big|_0^{t-3.5}$$

$$= \frac{e^3}{18}\left(1 - e^{-3(t-3.5)}\right)$$

Case 3: $t > 4$

$$x(t) * h_2(t) = \frac{1}{6}\int_0^{1/2} e^{-3(\tau-1)}\,d\tau$$

$$= \frac{e^3}{6}\int_0^{1/2} e^{-3\tau}\,d\tau$$

$$= -\frac{e^3}{18} e^{-3\tau}\Big|_0^{1/2}$$

$$= \frac{e^3}{18}\left(1 - e^{-3/2}\right)$$

Hence,

$$x(t) * h_2(t) = \begin{cases} 0 & , \quad t < 3.5 \\ \frac{e^3}{18}\left(1 - e^{-3(t-3.5)}\right) & , \quad 3.5 \le t \le 4 \\ \frac{e^3}{18}\left(1 - e^{-3/2}\right) & , \quad t > 4 \end{cases}$$

Finally, $x(t) * h(t) = x(t) h_1(t) - x(t) h_2(t)$
with $x(t) h_1(t)$ and $x(t) * h_2(t)$
given above.

b) Consider each term separately

$$\lim_{t \to \infty} x(t) * h_1(t)$$

$$= \lim_{t \to \infty} \frac{\pi}{3} e^{-3(t-3.5)} \left[u(t-2.5) - u(t-3)\right]$$

$$= 0$$

$$\lim_{t \to \infty} x(t) * h_2(t)$$

$$= \frac{e^3}{18}\left(1 - e^{-3/2}\right)$$

Hence

$$\lim_{t \to \infty} x(t) * h(t)$$

$$= -\frac{e^3}{18}\left(1 - e^{-3/2}\right)$$

9.4 a) Set $x(t) = u(t)$ and solve the integral.

$$y(t) = \int_2^4 e^{2\tau-6}\, u(t-\tau)\, d\tau$$

Case 1: $t < 2$ $\qquad y(t) = 0$

Case 2: $2 \le t \le 4$

$$y(t) = \int_2^t e^{2\tau-6}\, d\tau$$

$$= e^{-6} \int_2^t e^{2\tau}\, d\tau$$

$$= e^{-6} \frac{1}{2} e^{2\tau} \Big|_2^t$$

$$= \frac{e^{-6}}{2} \left(e^{2t} - e^{4}\right)$$

Case 3: $t > 4$

$$y(t) = \int_2^4 e^{2\tau-6}\, d\tau$$

$$= e^{-6} \frac{1}{2} e^{2\tau} \Big|_2^4$$

$$= e^{-6} \frac{1}{2} \left(e^{8} - e^{4}\right)$$

$$= \frac{e^{2} - e^{-2}}{2}$$

Hence, step response is

$$y(t) = \begin{cases} 0 & , \quad t < 2 \\ \frac{e^{-6}}{2}(e^{2t} - e^{4}) & , \quad 2 \le t \le 4 \\ \frac{e^{2} - e^{-2}}{2} & , \quad t > 4 \end{cases}$$

b) Set $x(t) = \delta(t)$ and solve the integral

$$h(t) = \int_{2}^{4} e^{2\tau - 6}\,\delta(t-\tau)\,d\tau$$

Case 1: $t < 2$, $h(t) = 0$

Case 2: $2 \le t \le 4$, $h(t) = e^{2t-6}$

Case 3: $t > 4$, $h(t) = 0$

From part a), we have

$$\frac{dy(t)}{dt} = \begin{cases} 0 & , \quad t < 2 \\ e^{2t-6} & , \quad 2 \le t \le 4 \\ 0 & , \quad t > 4 \end{cases}$$

$\therefore$ impulse resp. is derivative of step resp.

c) Test for bounded-input bounded-output stability.

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{2}^{4} e^{2t-6}\,dt$$

$$= e^{-6}\, \tfrac{1}{2} e^{2t} \Big|_{2}^{4}$$

$$= \frac{e^{-6}}{2}\left(e^{8} - e^{4}\right)$$

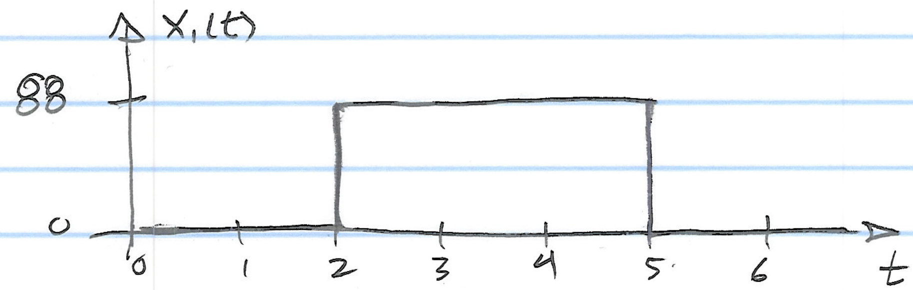
$$= \frac{e^{2} - e^{-2}}{2} < \infty$$

$\therefore$ system is stable

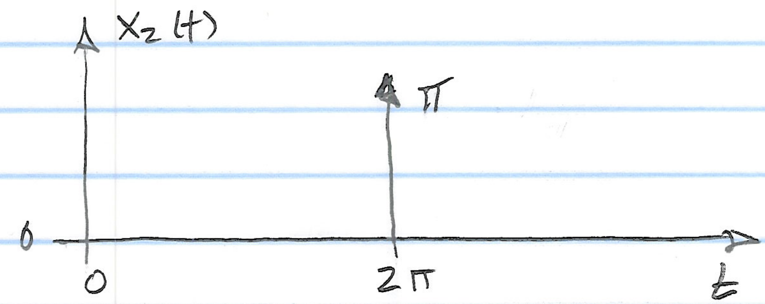
d) System is causal if
$h(t) = 0, \quad t < 0$

From part b), this is clearly the case, since $h(t) = 0, \quad t < 2$

9.5a) straightforward.

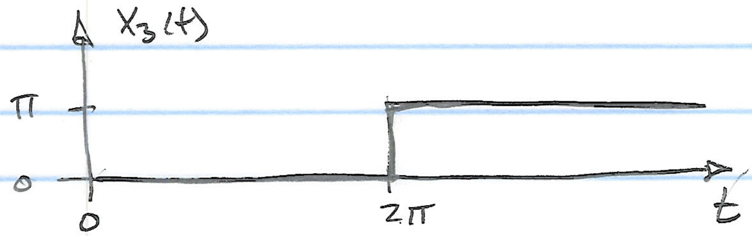


b) straightforward



c) Integral of impulse gives a step

$x_3(t) = \pi\, u(t - 2\pi)$



d) shift right $e^{-t}u(t)$ by 4.