Problem Set # 5

ECE 2025

SOLUTIONS

5.1 Spectrograms of two periodic signals $x(t)$ and $y(t)$ are shown.

(a) The spectrum line at $f = 2000$ Hz is clearly the 3rd harmonic or $N = 3$, since the fundamental is at $\frac{2000}{3}$ Hz. ////

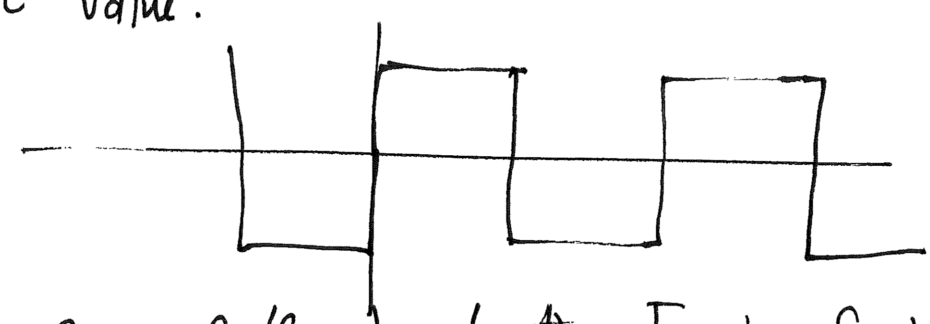
Note: The second harmonic of $y(t)$ does not have any energy.

(b) Both signals have the same fundamental frequency.

$$f_0 = \frac{2000}{3} \text{ Hz} \Rightarrow T_0 = \frac{3}{2000} \text{ Seconds.}$$ ////

(c) One of the signals is a 50% duty-cycle square wave. See Page 52 of the textbook.

We also assume that the square wave does not have a DC value.



The Even Coefficient of the Fourier Series is ZERO. This corresponds to $y(t)$.

Problem 5.2

Sketch the spectrum for the signals defined in Matlab.
Label all frequencies and complex amplitudes.

(a)

$$zz = -29 + \sum_{-2}^{2} 13jk \, \exp^{j55\pi kt}$$

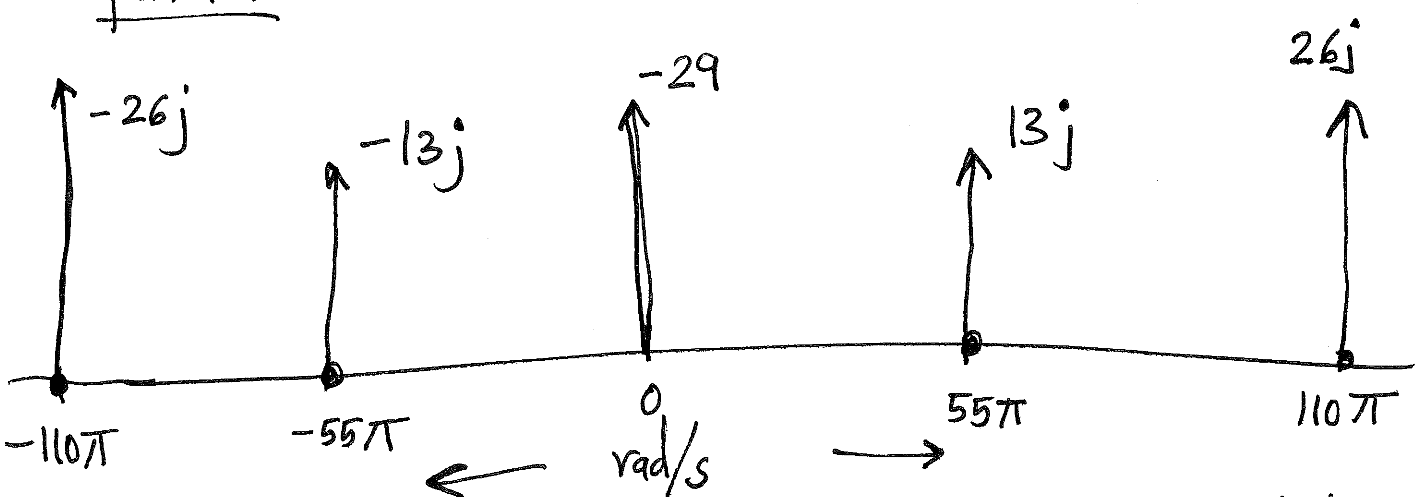
i.e.

$$zz(t) = -29 + \sum_{k=-2}^{2} 13jk \, \exp^{j55\pi kt}$$

Fundamental $\omega_0 = 55\pi$

$$zz(t) = -29 + 13j(-2)e^{j55\pi(-2)t} + 13j(-1)e^{j55\pi(-1)t} + 13j(0)e^{j55\pi \cdot 0 \cdot t} + 13j(1)e^{j55\pi(1)t} + 13j(2)e^{j55\pi(2)t}$$

Spectrum



spectrum of (a)

Problem 5.2

(b)

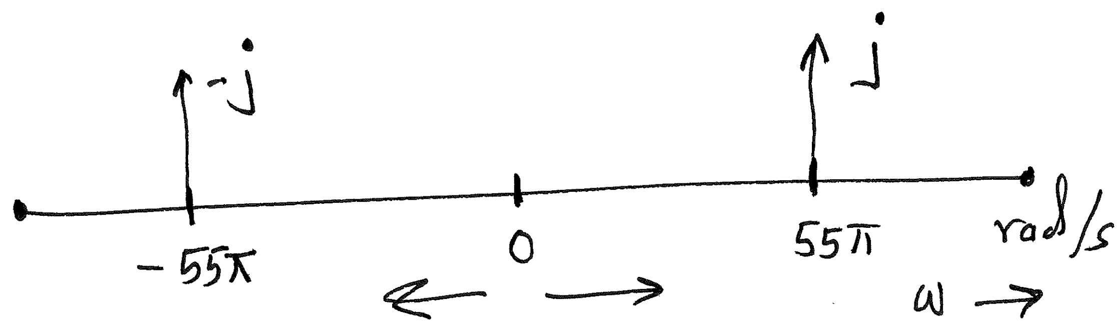
$$zx(t) = \sum_{k=-1}^{1} jk\, e^{j55\pi k t}$$

and

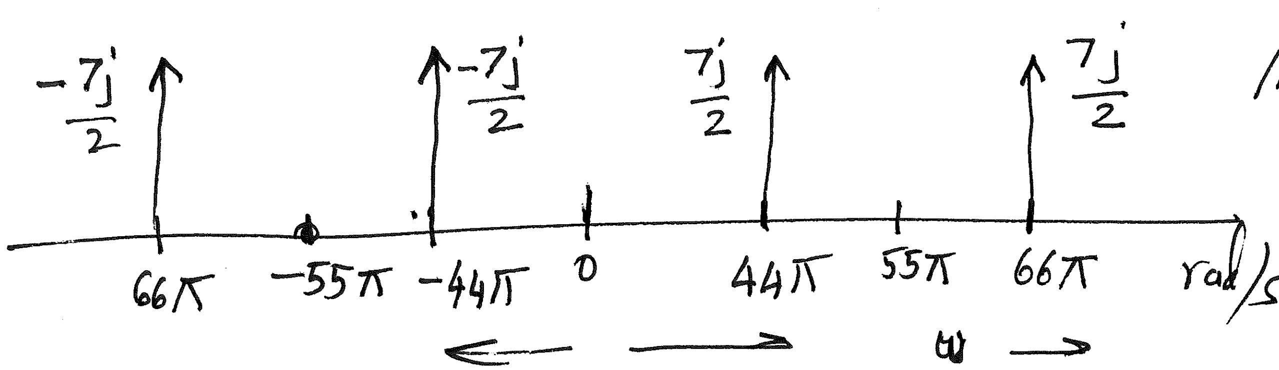
$$x(t) = 7\cos(11\pi t)\left[\sum_{k=-1}^{k=1} jk\, e^{j55\pi k t}\right]$$

$$\therefore \quad x(t) = 7\underbrace{\left[\frac{e^{j11\pi t}}{2} + \frac{e^{-j11\pi t}}{2}\right]}_{a(t)}\underbrace{\left[\sum_{k=-1}^{1} jk\, e^{j55\pi k t}\right]}_{z(t)}$$

spectrum of $z(t)$ is



Therefore, the spectrum of $x(t) = a(t) * z(t)$



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Problem 5.3

A periodic signal $x(t)$ is represented as a Fourier Series

$$x(t) = 10 + \sum_{k=-\infty}^{\infty} k^2 e^{j32\pi kt}$$

(a) $\omega_0 = 32\pi$, $T_0 = \frac{2\pi}{\omega_0} \Rightarrow T_0 = \frac{1}{16}$ seconds

(b) $$y(t) = 10 + \sum_{k=-\infty}^{\infty} k^2 e^{j32\pi kt} + \frac{9}{2} e^{j\frac{\pi}{2}} e^{j96\pi t} + \frac{9}{2} e^{-j\frac{\pi}{2}} e^{-j96\pi t}$$

$y(t)$ is to written in the form

$$y(t) = \sum_{k} b_k e^{j32\pi kt}$$

The two exponential terms are added only when $k = \pm 3$.

Therefore, $b_k = a_k$, except when $k = \pm 3$.

$$b_3 = a_3 + \frac{9}{2} e^{j\pi/2}, \quad b_{-3} = a_{-3} + \frac{9}{2} e^{-j\frac{\pi}{2}}.$$

$x(t)$

a_k	MAG	PHASE
a_3	9	0
a_2	4	0
a_1	1	0
a_0	10	0
a_{-1}	1	0
a_{-2}	4	0
a_{-3}	9	0

$y(t)$

b_k	MAG	PHASE rads
b_3	$4.5\sqrt{5}$	0.464
b_2	same	same
b_1	same	same
b_0	same	same
b_{-1}	same	same
b_{-2}	same	same
b_{-3}	$4.5\sqrt{5}$	-0.464

Problem 5.4

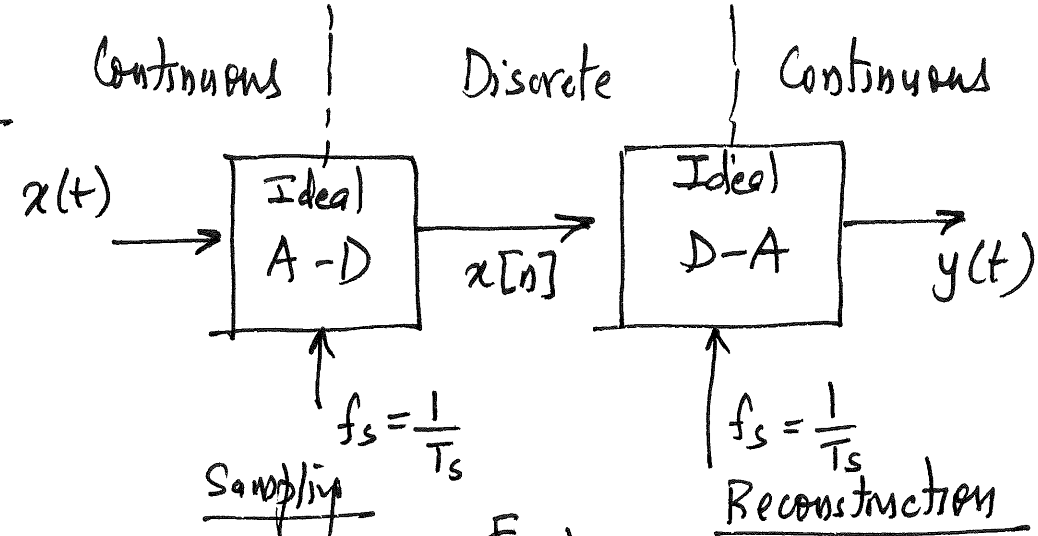
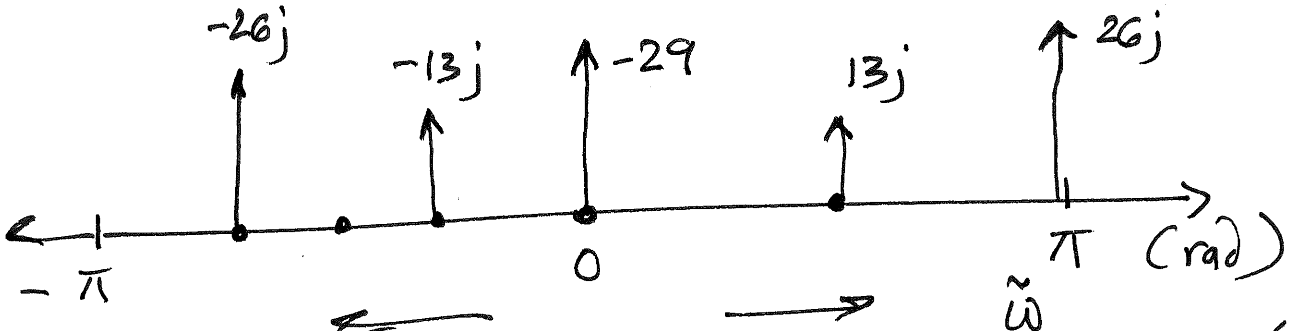


Fig)

(a) Let $x[n] = -29 + \sum_{\ell=-2}^{2} j\,13\,\ell\, e^{j0.3\pi \ell n}$

Plot spectrum for discrete-time signal $x[n]$ over $-\pi \leq \tilde{\omega} \leq \pi$

$$x[n] = -29 + \left[-13j\, e^{-j0.3\pi n} - 26j\, e^{-j0.6\pi n} + 26j\, e^{j0.6\pi n} + 13j\, e^{j0.3\pi n} \right]$$



(b) Let the discrete-time signal $x[n]$ be given by

$$x[n] = 9.87 \cos\left[0.3\pi(n-1)\right] \quad \cdots \text{(A)}$$

let the sampling rate $f_s = 1500$ samples/s.

* Find two possible input signals $x_1(t)$ and $x_2(t)$ that can result in the same $x[n]$ of (A)

The objective is to find aliases of

$$x[n] = 9.87 \cos(0.3\pi(n-1))$$

that have a frequency between 0 and 1500 Hz when sampled at 1500 samples/sec.

(i) We can reconstruct $x[n]$ with the given f_s

The first alias @ 225 Hz

$$x_1(t) = 9.87 \cos(0.3\pi(t-1))$$

$$= 9.87 \cos(0.3\pi f_s t - 0.3\pi)$$

$$= 9.87 \cos(0.3\pi(1500)t - 0.3\pi)$$

$$= 9.87 \cos(450\pi t - 0.3\pi)$$

(ii) Now, we should look for the second alias.

The second alias can be found by subtracting the argument of the cosine from an integer multiple of 2π.

Recall $x[n] = 9.87 \cos(0.3\pi(n-1))$

rewrite it as

$$x[n] = 9.87 \cos\left[\underbrace{2\pi n - 1.7\pi n}_{(= 0.3\pi n)} - 0.3\pi\right]$$

$$= 9.87 \cos\left[2\pi n - (1.7\pi n + 0.3\pi)\right]$$

$$= 9.87 \cos(1.7\pi n + 0.3\pi) \quad \cdots \text{(B)}$$

Reconstructing (B) with $f_s = 1500$ Samples/second

$$x_2(t) = 9.87 \cos(1.7\pi f_s t + 0.3\pi)$$

$$= 9.87 \cos((1.7 \times 1500)\pi t + 0.3\pi)$$

$$= 9.87 \cos(2550\pi t + 0.3\pi)$$

second alias @ 1275 Hz

The two aliases are

$$x_1(t) = 9.87 \cos(450\pi t - 0.3\pi)$$

$$x_2(t) = 9.87 \cos(2550\pi t + 0.3\pi)$$

both $x_1(t)$ and $x_2(t)$, when sampled at $f_s = 1500$ Hz result in

$$x[n] = 9.87 \cos(0.3\pi(n-1))$$

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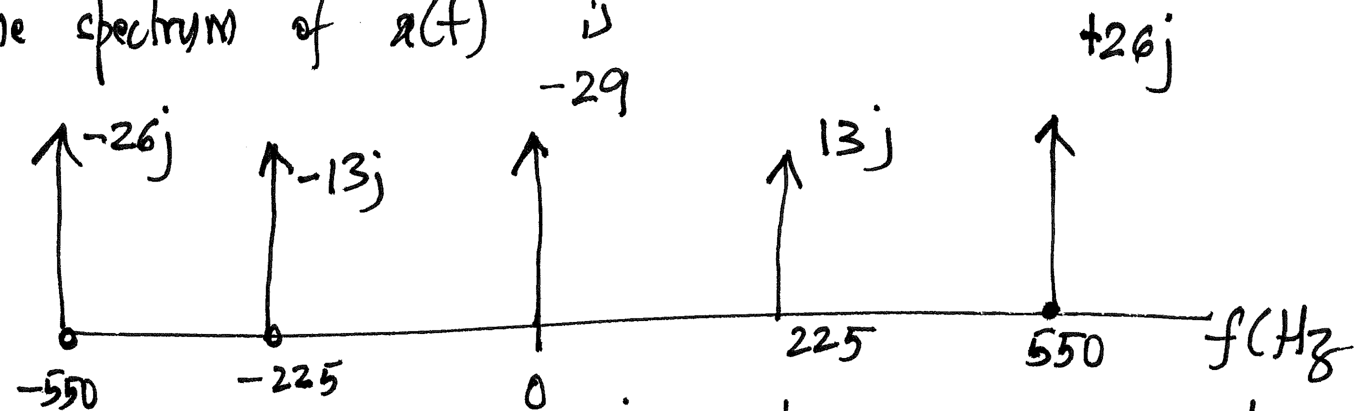
Problem 5.5

$$x(t) = -29 + \sum_{\ell=-2}^{2} j\,13\ell\, e^{j550\pi \ell t}$$

(a) Determine the minimum sampling rate f_s, so that output $y(t)$ is equal to input $x(t)$

The spectrum of $x(t)$ is



$$x(t) = -29 + j\,13(-2)\,e^{-j1100\pi t} + j\,13(-1)\,e^{-j550\pi t}$$
$$+ j\,13(1)\,e^{+j550\pi t} + j(+2)13\cdot e^{+j1100\pi t}$$

$$\Rightarrow f_s > 2 * f_{max}$$

$$\Rightarrow f_s > 2 \times 550$$

$$\Rightarrow f_s > 1100 \text{ Hz}.$$

(b) If we sample at 1000 Hz $= f_s$ then we have the discrete time signal $x[n]$ as

5.5 (b) Continued

Note: we are sampling at $f_s = \underline{1000\ Hz}$

$$x[t] = -29 + 13j\left(e^{j550\pi t} - e^{-j550\pi t}\right)$$

$$+ 26j\left(e^{j1100\pi t} - e^{-j1100\pi t}\right)$$

$$= -29 + (13j \cdot 2j)\left[\sin 550\pi t\right]$$

$$+ (26j \cdot 2j)\left[\sin 1100\pi t\right]$$

$$= -29 + (-26)\sin 550\pi t + (-52)\sin 1100\pi t$$

$$= -29 + (-26)\cos\left(\pi 550 t - \frac{\pi}{2}\right)$$

$$- 52\cos\left(1100\pi t - \frac{\pi}{2}\right)$$

$$\therefore x(t) = -29 - 26\cos\left(550\pi t - \frac{\pi}{2}\right) - 52\cos\left(1100\pi t - \frac{\pi}{2}\right)$$

$$x[n] = -29 - 26\cos\left(2\pi\left(\frac{225}{1000}\right)n - \frac{\pi}{2}\right)$$

$$- 52\cos\left(2\pi\left(\frac{550}{1000}\right)n - \frac{\pi}{2}\right)$$

$$\therefore x[n] = -29 - 26\cos\left(0.55\pi n - \frac{\pi}{2}\right)$$

$$- 52\cos\left(1.1\pi n - \frac{\pi}{2}\right) \quad ///$$

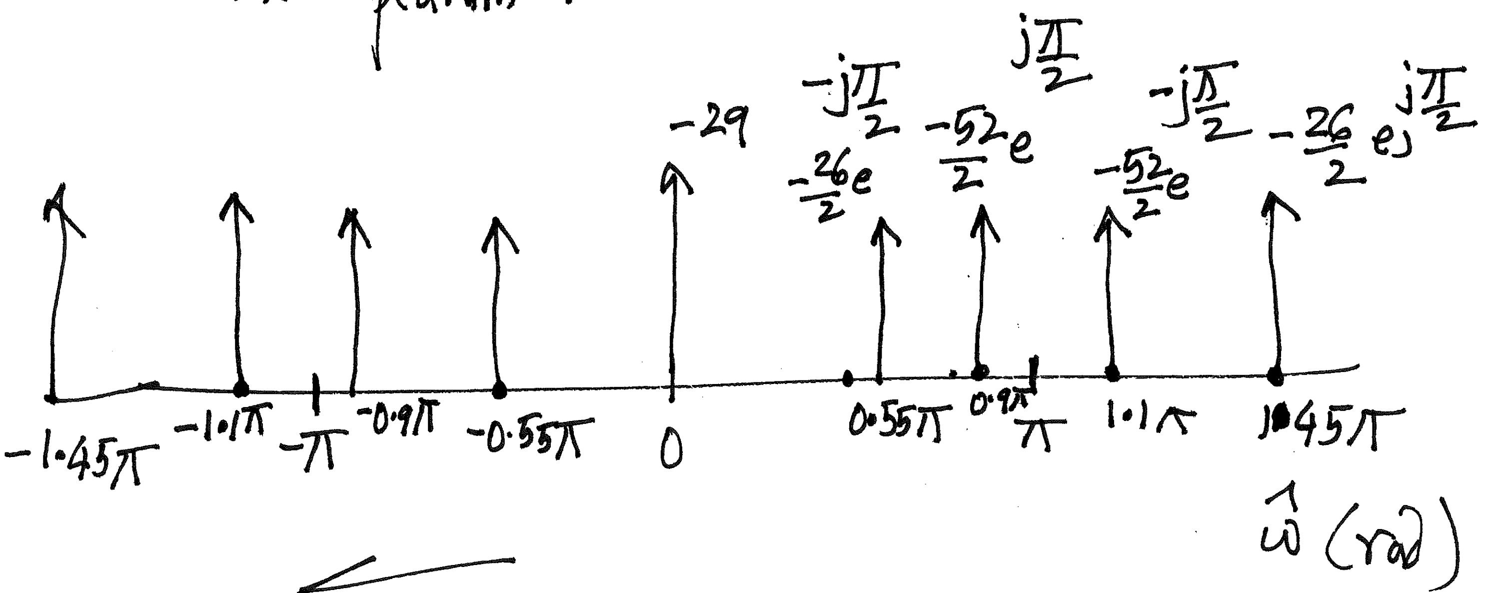
5.5 (b) continued

We can also have the aliases below

$$x_2[n] = -29 - 26\cos\left(2\pi n - \left(1.45\pi n + \frac{\pi}{2}\right)\right)$$
$$- 52\cos\left(2\pi n - \left(0.9\pi n + \frac{\pi}{2}\right)\right)$$
$$= -29 - 26\cos\left(2\pi n - \left(1.45\pi n + \frac{\pi}{2}\right)\right)$$
$$- 52\cos\left(2\pi n - \left(0.9\pi n + \frac{\pi}{2}\right)\right)$$
$$= -29 - 26\cos\left(1.45\pi n + \frac{\pi}{2}\right)$$
$$- 52\cos\left(0.9\pi n + \frac{\pi}{2}\right)$$

////

The spectrum is



$$e^{j\pi/2} = \underbrace{\cos\frac{\pi}{2}}_{=0} + j\underbrace{\sin\frac{\pi}{2}}_{=1} = j$$

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Problem 5.6(a)

```
nn = 0: 2301832 ;
xx = cos( 2*pi*0.75*nn +5)
soundsc (x, 8000) ;
```

$$xx[n] = \cos(2\pi(0.75)n + 5)$$
$$= \cos(1.5\pi n + 5)$$
$$= \cos((2\pi n - 0.5\pi n) + 5)$$
$$= \cos(2\pi n - (0.5\pi n - 5))$$

principal alias $\longrightarrow$ $= \cos(0.5\pi n - 5)$ ////

When reconstructed with a sampling rate of 8000 Hz, the resulting analog signal is

$$xx(t) = \cos[(2\pi(0.25 * 8000)t - 5]$$
$$= \cos(2\pi(2000)t - 5)$$

The analog frequency is 2000 Hz. ////

(b) Matlab

```
tt = 0: 1/8000 : 10 000 ;
xx = (28*pi)*cos(2*pi*800*tt-7) ;
soundsc (xx, fsamp) ;
```

Find fsamp so that an analog frequency of 400 Hz may be heard.

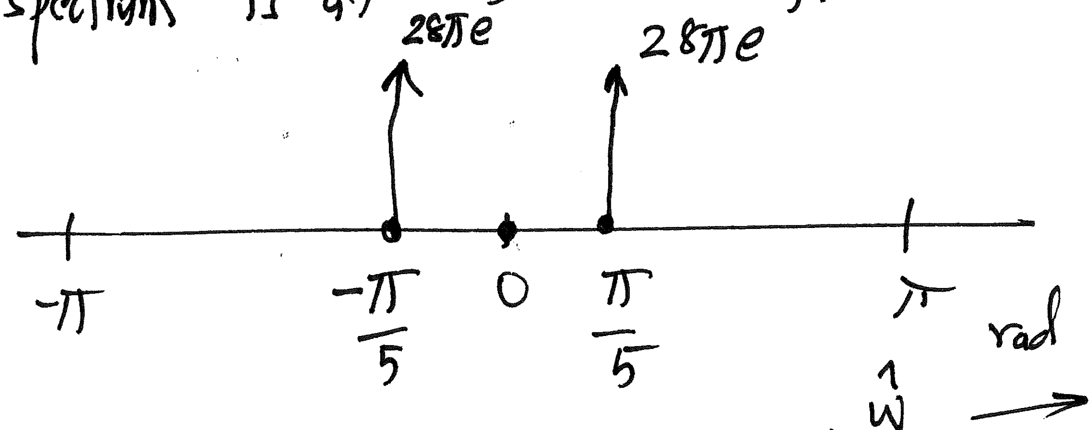
Ans: Note: $xx(t)$ is a continuous time signal that is sampled at 8000 samples/s to yield the following

$$xx[n] = 28\pi \cos\left(2\pi\left(\frac{800}{8000}\right)n - 7\right)$$

$$= \cancel{28\pi\left(\frac{2\pi}{10}n - 7\right)}$$

$$= 28\pi \cos\left(\frac{2\pi}{10}n - 7\right)$$

The spectrum is at



$$xx[n] = \left(28\pi \cos\left(2\pi\left(\frac{1}{10}\right)n - 7\right)\right) \quad \ldots \text{(c)}$$

To generate an analog signal of 400 Hz, the signal should be reconstructed as a frequency of

$$f = \frac{400}{0.1} \quad (\text{from } c)$$

$$= 4000 \text{ Hz}$$

(c) **Matlab Code**

$$tt = 0 : \frac{1}{8000} : 40 ;$$

```
xx = cos ( 2*pi * 7000 * tt ) ;
Soundsc (xx, 20000);
```

Determine the frequency (in Hz) and duration (sec) of the final played tone.

Ans: Original tone has length = 40 seconds.

The sampling rate = 8000 Samples/s

The total number of samples = 8000 × 40

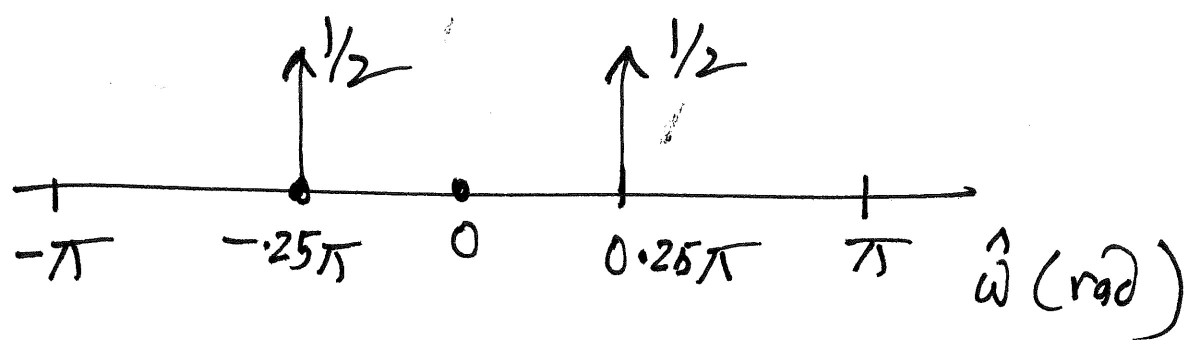
= 32,0000

= 320,000 samples

$$x[n] = \cos\left(2\pi \cdot \frac{7000}{8000} n\right)$$ is found as follows.

10

$$x[n] = \cos\left(2\pi \cdot \frac{7000}{8000} n\right)$$
$$= \cos\left(2\pi \cdot \frac{7}{8} \cdot n\right)$$
$$= \cos(1.75\pi n)$$
$$= \cos((2\pi - 0.25\pi) n)$$
$$= \cos(2\pi n - 0.25\pi n)$$
$$= \cos(0.25\pi n)$$



When reconstructed at 20,000 Hz by soundsc(xx, 20,000) we get

$$y(t) = \cancel{\cos(0.2\pi A}$$
$$\cos(0.25\pi(20000)t)$$
$$\boxed{y(t) = \cos(2\pi(2500)t)}$$

The final duration when reconstructed at 20,000 Hz is $t = \dfrac{320{,}000 \text{ samples}}{20{,}000 \text{ samples/second}} = 16$ seconds