SOLUTIONS TO ECE 2025 FALL 2009 PROBLEM SET #5

PROBLEM 5.1*:

Approach:

(a) The equation describes one period; replicating this segment gives the periodic signal

(b) The given signal is a scaled and stretched version of a signal whose Fourier series coefficients are already known; we will use the scaling property to find the Fourier series of the given signal.

(c) Integrate the signal using the formula for the area of a triangle

(d) The spectrum of a periodic signal has lines at integer multiples of the fundamental frequency, with weights given by the Fourier series coefficients.

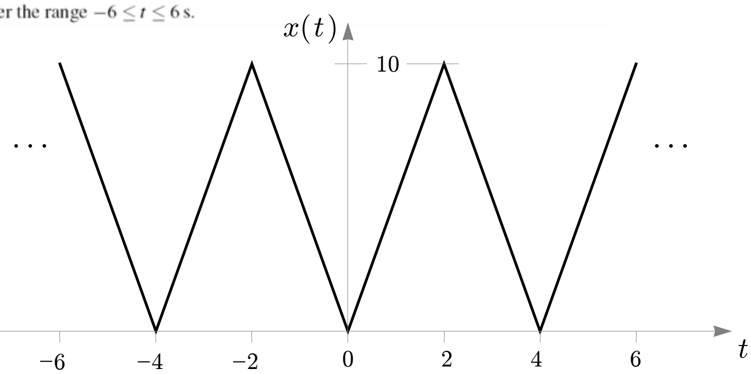
(e) Use Euler to combine the positive- & negative-frequency exponentials into a single cosine

Suppose that a periodic signal has a period of 4 s, and is defined *over one period* as:

$$x(t) = 5|t| \quad \text{for } -2 \le t \le 2$$

Then this signal is actually a scaled version of the triangle wave in Section 3-6.4 (page 55) of *SP-First*.

(a) Draw a plot of $x(t)$ over the range $-6 \le t \le 6$ s.



(b) Determine a general expression for all the Fourier series coefficients, a_k. No integrals should be required if you use the Fourier series information given in equation (3.39) on p. 56.
Hint: exploit the *amplitude scaling property* which says that the Fourier coefficients of $Ax(t)$ are Aa_k, where $\{a_k\}$ are the Fourier coefficients of $x(t)$.

The above triangle wave $x(t)$ has the same *shape* as that of Fig. 3-18 on page 55 of the book, let's call it $s(t)$, but with a different period and amplitude. This means we can write

$$x(t) = cs(dt)$$

for some constants c and d, namely $c = 10$ and $d = 0.1$.
From this we can see that the Fourier series coefficients for $x(t)$ are simply related to those of $s(t)$, for if $s(t) = \sum_k a_k e^{jk2\pi t/T_0}$, then

$$\begin{aligned} x(t) &= cs(dt) \\ &= c\sum_k a_k e^{jk2\pi(dt)/T_0} \\ &= \sum_k a_k' e^{jk2\pi t/T_0'} \end{aligned}$$

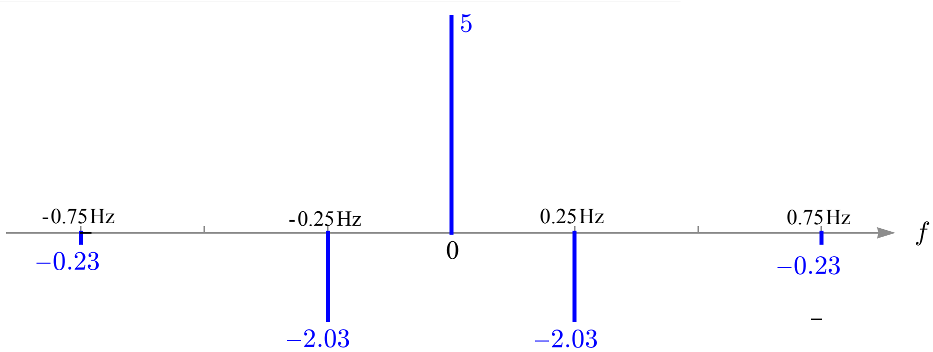
has Fourier series coefficients $a_k' = ca_k$ and period $T_0' = T_0/d$. We conclude that stretching the signal horizontally (along the time axis) has no impact on the Fourier series coefficients; it merely changes the fundamental period. And scaling the amplitude of a periodic signal scales the Fourier series coefficients by the same amount. Scaling the Fourier series coefficients from (3.39) by $c = 10$ yields:

$$a_k' = \begin{cases} \dfrac{-20}{\pi^2 k^2} & k = \pm1,\ \pm3,\ \pm5,\ \ldots \\ 0 & k = \pm2,\ \pm4,\ \pm6,\ \ldots \\ 5 & k = 0 \end{cases}.$$

(c) Determine the DC value by integrating $x(t)$ over one period, and dividing by the period.

$$a_0 = \frac{1}{T}\int_{-0.5T}^{0.5T} x(t)dt = \frac{1}{T}(\text{area under triangle}) = \frac{1}{4}(0.5)(\text{base})(\text{height}) = \frac{1}{4}(0.5)(4)(10) = 5.$$

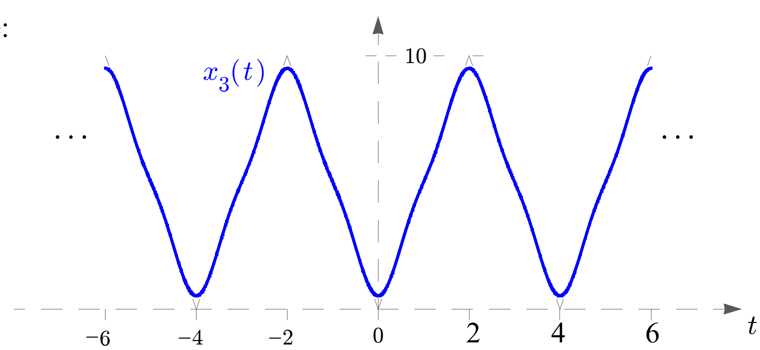
(d) Make a spectrum plot of the triangle-wave signal defined above, showing a frequency range in hertz that includes the first three harmonics, i.e., $k = 0, \pm1, \pm2, \pm3$.



(e) Define $x_3(t)$ as the signal synthesized from the first three harmonics (plus DC). Write a mathematical formula for $x_3(t)$ as a constant plus sinusoidal terms.
Note: you can check your work on this problem and the previous problem by using MATLAB to plot $x_3(t)$ for two or three periods—you should get a close approximation to a triangle wave.

$$\begin{aligned} x_3(t) &= \textstyle\sum_{k=-3}^{3} a_k' e^{jk2\pi t/T_0'} \\ &= a_{-3}' e^{-j6\pi t/T_0'} + a_{-1}' e^{-j2\pi t/T_0'} + a_0' + a_1' e^{j2\pi t/T_0'} + a_3' e^{j6\pi t/T_0'} \\ &= a_0' + \underbrace{a_1'(e^{j2\pi t/T_0'} + e^{-j2\pi t/T_0'})}_{\text{[because } a_{-1}' = a_1' \text{]}} + \underbrace{a_3'(e^{j6\pi t/T_0'} + e^{-j6\pi t/T_0'})}_{\text{[because } a_{-3}' = a_3' \text{]}} \\ &= a_0' + 2a_1' \cos(2\pi t/T_0') + 2a_3' \cos(6\pi t/T_0') \\ &= 5 - \frac{40}{\pi^2}\cos(0.5\pi t) - \frac{40}{9\pi^2}\cos(1.5\pi t), \end{aligned}$$

as shown here:



PROBLEM 5.2*:

Approach:

(a) The period can be found by inspection: the given signal is expanded in terms of its Fourier series

(b) Adding a sinusoid whose frequency is an integer multiple of the fundamental frequency will only change two of the Fourier series coefficients; the others will remain the same.

A periodic signal $x(t)$ is represented as a Fourier series of the form

$$x(t) = -2 + \sum_{k=-\infty}^{\infty} k^2 e^{j150\pi kt}$$

(a) Determine the fundamental **period** of the signal $x(t)$, i.e., the minimum period. Explain.

The given equation has the form $\sum_k a_k e^{jk2\pi t/T_0}$ where $a_0 = -2$, $a_k = k^2$ for nonzero k, and $T_0 = 1/75$.

The fundamental period is therefore $T_0 = 1/75$.

A periodic signal $x(t)$ is represented as a Fourier series of the form

$$x(t) = -2 + \sum_{k=-\infty}^{\infty} k^2 e^{j150\pi kt}$$

(a) Determine the fundamental **period** of the signal $x(t)$, i.e., the minimum period. Explain.

(b) Define a new signal by adding a sinusoid to $x(t)$

$$y(t) = 3\sin(300\pi t) + x(t)$$

The new signal, $y(t)$ can be expressed in the following Fourier Series with new coefficients $\{b_k\}$:

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j150\pi kt}$$

Fill in the following tables, giving *numerical values* for each $\{a_k\}$ and $\{b_k\}$ in polar form.
Note: A magnitude value must be nonnegative.
Hint: Find a simple relationship between $\{b_k\}$ and $\{a_k\}$, where $\{a_k\}$ denotes the Fourier coefficients of the original signal $x(t)$.

Applying inverse Euler, the added term $3\sin(300\pi t)$ can be written as:

$$\begin{aligned} 3\sin(300\pi t) &= \frac{3}{2j} e^{j300\pi t} - \frac{3}{2j} e^{-j300\pi t} \\ &= \frac{3}{2j} e^{j(2)150\pi t} - \frac{3}{2j} e^{j(-2)150\pi t}. \end{aligned}$$

Adding this to $\sum_k a_k e^{jk150\pi t}$ will only impact the $k = \pm 2$ terms, the other terms in the sum will be unaffected. Therefore, we have

$$\begin{aligned} b_{-2} &= a_{-2} - 3/(2j) \\ b_2 &= a_2 + 3/(2j) \end{aligned}$$

and otherwise $b_k = a_k$ for all $k \neq \pm 2$.

Writing b_{-2} and b_2 in polar form yields:

$$\begin{aligned} b_{-2} = a_{-2} - 3/(2j) &= (-2)^2 - 3/(2j) \\ &= 4 + 1.5j \\ &= 4.272e^{j0.114\pi}, \\ b_2 = a_2 + 3/(2j) &= (2)^2 + 3/(2j) \\ &= 4 - 1.5j \\ &= 4.272e^{-j0.114\pi}. \end{aligned}$$

Filling in the table yields:

Signal: $x(t)$

a_k	Mag	Phase
a_3	9	0
a_2	4	0
a_1	1	0
a_0	2	π
a_{-1}	1	0
a_{-2}	4	0
a_{-3}	9	0

Signal: $y(t)$

b_k	Mag	Phase
b_3	*SAME*	
b_2	4.272	-0.114π
b_1	*SAME*	
b_0	*SAME*	
b_{-1}	*SAME*	
b_{-2}	4.272	0.114π
b_{-3}	*SAME*	

PROBLEM 5.3*:

Approach:

(a) The number of samples per period is the ratio of sampling frequency to fundamental frequency.
(b) Adding any integer multiple of the sampling frequency to the sinusoid frequency will not change its samples.
(c) As in (a): the no. samples per period is the ratio of sampling frequency to fundamental frequency.

Consider the cosine wave

$$x(t) = 10\cos(880\pi t + \phi)$$

Suppose that we obtain a sequence of numbers by sampling the waveform at equally spaced time instants nT_s. In this case, the resulting sequence would have values

$$x[n] = x(nT_s) = 10\cos(880\pi nT_s + \phi)$$

for $-\infty < n < \infty$. Suppose that $T_s = 0.0001$ sec.

(a) How many samples will be taken in one period of the cosine wave?

Roughly speaking, there will be 22.72 samples on *average* per period, since the ratio of the sampling frequency to the sinusoid frequency is

$$f_S/f_0 = 10000/440 = 22.7272\ldots$$

However, this is not the question that was asked; a precise answer to the question is that there will be either 22 or 23 samples in one period of the cosine wave, depending on both the value of the phase ϕ and on which "period of the cosine wave" is being considered. For example, there will be 23 samples in the period that begins and ends at a peak when $\phi = 0$, but there will be only 22 samples in that same period when $\phi = -\pi/100$.

(b) Now consider another waveform $y(t)$ such that

$$y(t) = 10\cos(\omega_0 t + \phi)$$

Find a frequency $\omega_0 > 880\pi$ such that $y(nT_s) = x(nT_s)$ for all integers n.
Hint: Use the fact that $\cos(\theta + 2\pi n) = \cos(\theta)$ if n is an integer.

Start with $x(nT_s) = 10\cos(880\pi nT_s + \phi)$ and then apply the hint by adding $2\pi n\ell$:

$$\begin{aligned} x(nT_s) &= 10\cos(880\pi nT_s + \phi) \\ &= 10\cos(880\pi nT_s + 2\pi n\ell + \phi) \\ &= 10\cos((880\pi + \ell 2\pi f_s)nT_s + \phi) \end{aligned}$$

This last equation is a sampled version of $10\cos(\omega_0 t + \phi)$, where

$$\omega_0 = 880\pi + \ell 2\pi f_s.$$

Any integer $\ell \geq 1$ is acceptable. Intuitively this equation states that the sampled signal does not change whenever the sinusiod frequency is increased or decreased by an integer multiple of the sampling frequency. For example, choosing $\ell = 1$ yields

$$\omega_0 = 880\pi + 20000\pi = 20880\pi.$$

(c) For the frequency found in (b), what is the total number of samples taken in one period of $x(t)$?

There will be $f_s/f_0 = 10000/10440 = 0.958$ samples on average per period. In any given period, the number of samples will be either zero or one, depending on the phase and period boundaries.

PROBLEM 5.4*:

Approach:

Same as Prob. 5.3: Adding any integer multiple of the sampling frequency to the sinusoid frequency will not change its sampled values.

Suppose that a discrete-time signal $x[n]$ is given by the formula

$$x[n] = 2.2\cos(0.3\pi n - \pi/3)$$

and that it was obtained by sampling a continuous-time signal $x(t) = A\cos(2\pi f_0 t + \phi)$ at a sampling rate of $f_s = 6000$ samples/sec. Determine three different continuous-time signals that could have produced $x[n]$. All these continuous-time signals should have a frequency less than 8 kHz. Write the mathematical formula for all three.

On the one hand, we can add $\ell 2\pi n$ to the argument of the cosine without changing anything, so that:

$$x[n] = 2.2\cos((0.3\pi + \ell 2\pi)n - \pi/3)$$

for any integer ℓ.

On the other hand, sampling $x(t) = A\cos(2\pi f_0 t + \phi)$ at a sample rate of $f_s = 6000\,\text{Hz}$ yields

$$x[n] = A\cos((2\pi f_0/6000)n + \phi).$$

Equating like terms, we conclude that $A = 2.2$ and $\phi = -\pi/3$, and further that:

$$0.3\pi + \ell 2\pi = 2\pi f_0/6000 \qquad \Rightarrow \qquad f_0 = 900 + \ell 6000 \text{ Hz}.$$

Choosing $\ell \in \{-1, 0, 1\}$ yields three possible continuous-time signals:

$$\begin{array}{lllll} \ell = -1 & \Rightarrow & f_0 = -5100 & \Rightarrow & x(t) = 2.2\cos(10200\pi t + \pi/3) \\ \ell = 0 & \Rightarrow & f_0 = 900 & \Rightarrow & x(t) = 2.2\cos(1800\pi t - \pi/3) \\ \ell = 1 & \Rightarrow & f_0 = 6900 & \Rightarrow & x(t) = 2.2\cos(13800\pi t - \pi/3), \end{array}$$

all of which have a frequency of less than 8 kHz.

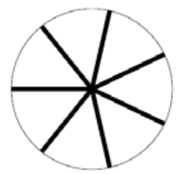
PROBLEM 5.5*:

Approach:

First translate the “appears stationary” condition to a condition on the rotational frequency of the wheel; second translate this frequency to the car speed using the fact that the wheel rotates once each time the car traverses a distance equal to the wheel diameter; third solve for the car speed.

When watching old TV movies, all of us have seen the phenomenon where a wagon wheel appears to move backwards. The same illusion can also be seen in automobile commercials, when the hubcaps of a car or truck have a spoked pattern. Both of these are due to the fact that the video consists of a sequence of samples (frames) of the scene. For this problem, assume that the frame sampling rate is 25 frames per second (the PAL standard used in Europe).

In the figure to the right, a seven-spoked wheel is shown. Assume that the diameter of this wheel is 0.5 meters. In addition, assume that the wheel is rotating clockwise, so that if attached to a car, the car would be traveling to the right *at a constant speed.* However, when seen on TV the spoke pattern of the car wheel appears to stand still. How fast is the car traveling (in kilometers per hour)? Derive a general equation that will make it easy to give all possible answers.



An obvious reason that the wheel might appear stationary is because it *is* stationary, i.e., the car is not moving! So a speed of 0 km/h is going to be one valid answer. But another way the wheel might appear stationary is if the spoke pattern appears the same each time the video camera samples (i.e. takes a picture); this will happen when the sampling time is precisely an integer multiple of the time it takes the wheel to rotate by an angle of $2\pi/N$, where N is the number of spokes. In other words, if $T_s = 1/f_s$ is the sampling period and T_R is the time for one complete rotation, then the wheel will appear stationary whenever:

$$T_s = \ell T_R/N$$

for some integer ℓ. But the rotation period is related to the wheel diameter d and speed v by

$$T_R = \frac{\pi d}{v}.$$

Therefore, the wheel will appear stationary whenever:

$$T_s = \ell \frac{\pi d}{N v}$$

or equivalently $$v = \ell \frac{\pi d f_s}{N}.$$

for any integer ℓ. This is the general equation.

Substituting $d = 0.5$ m, $f_s = 25$ Hz, $N = 7$ spokes, and converting to km/h yields:

$$\begin{aligned} v &= \ell\pi(0.5 \text{ m})(25 \text{ Hz})/(7) \\ &= \ell\frac{12.5\pi}{7} \text{ m/s} \quad = \quad (\ell\frac{12.5\pi}{7} \text{ m/s})\frac{(1 \text{ km})/(1000 \text{ m})}{(1 \text{ hr})/(3600 \text{ s})} \\ &\qquad\qquad\qquad\;\; = \quad \ell\frac{45\pi}{7} \text{ km/hr} \\ &\qquad\qquad\qquad\;\; = \quad 20.196\ell \text{ km/hr} \end{aligned}$$

for any integer ℓ.