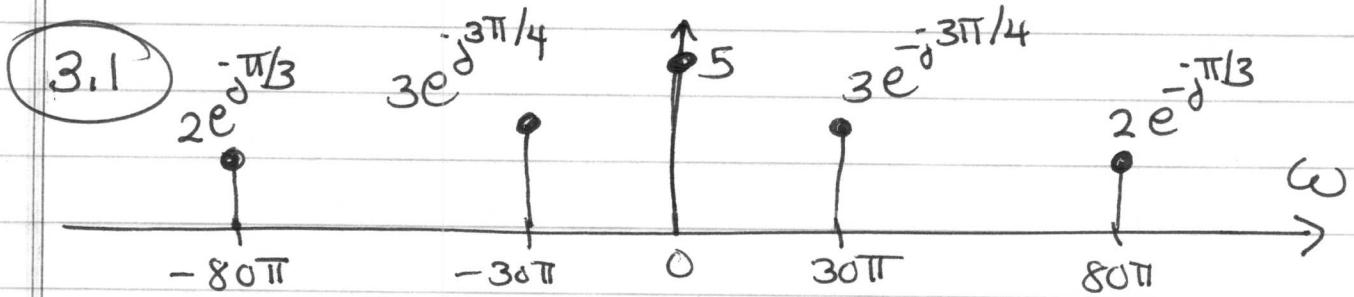


(1)

ECE 2025 FALL 2009

PROBLEM SET #3 - SOLUTION
 (prepared by MARK RICHARDS)



a) Each spectral line is the complex amplitude of a complex sinusoid of that frequency, so by inspection,

$$x(t) = 5e^{j0t} + 3e^{-j\frac{3\pi}{4}} e^{j30\pi t} + 3e^{+j\frac{3\pi}{4}} e^{-j30\pi t} \\ + 2e^{-j\frac{\pi}{3}} e^{j80\pi t} + 2e^{+j\frac{\pi}{3}} e^{-j80\pi t}$$

Combining exponents where possible, we get

$$x(t) = 5e^{j0t} + 3 \left(e^{j(30\pi t - \frac{3\pi}{4})} + e^{-j(30\pi t - \frac{3\pi}{4})} \right) \\ + 2 \left(e^{j(80\pi t - \frac{\pi}{3})} + e^{-j(80\pi t - \frac{\pi}{3})} \right)$$

The inverse Euler formula says $\cos \alpha = \frac{1}{2} (e^{j\alpha} + e^{-j\alpha})$;
 recognizing that we have a couple of groupings like this,

$$x(t) = 5 + 6 \cos(30\pi t - \frac{3\pi}{4}) + 4 \cos(80\pi t - \frac{\pi}{3})$$

(2)

Two Notes:

- ① Look for pairs of terms of the form $(e^{j\alpha} + e^{-j\alpha})$, or $(e^{j\alpha} - e^{-j\alpha})$. You can turn the first case into a cosine, the second into a sine.
- ② If $x(t)$ is a sum of cosines, the way we usually express it in 2025, is every cosine creates a pair of spectral lines, one at $+\omega_0$, the other at $-\omega_0$, and with complex amplitudes that are conjugates of one another. If your spectral lines occur in symmetric conjugate pairs as in this example, you can pretty much write down the cosines directly from the positive-frequency amplitudes:

$$x(t) = 5 + 2 \cdot 3 \cos(30\pi t - 3\pi/4) + 2 \cdot 2 \cos(80\pi t - \frac{\pi}{3})$$

[b] Remember that time delay is equivalent to a phase shift,
 $x(t) = A \cos(\omega t + \phi)$;
 $x(t-t_d) = A \cos(\omega(t-t_d) + \phi) = A \cos(\omega t + \phi - \omega t_d)$
So time delay decreases the phase by $-\omega t_d$.

We have $x(t)$ as a sum of three sinusoids, so three different ω 's & thus three different phase shifts,

$$\omega = 0 \text{ (DC term)}: \Delta\phi = -(0)(0.01) = 0$$

$$\omega = 30\pi: \Delta\phi = -(30\pi)(0.01) = 0.3\pi$$

$$\omega = 80\pi: \Delta\phi = -(80\pi)(0.01) = 0.8\pi$$

(3)

$$\text{So, } x(t) = 5 + 6 \cos(30\pi t - \frac{3\pi}{4} - 0.3\pi) + 2 \cos(80\pi t - \frac{\pi}{3} - 0.8\pi)$$

$$\Rightarrow x(t) = 5 + 6 \cos(30\pi t - \frac{21}{20}\pi) + 2 \cos(80\pi t - \frac{17}{15}\pi) \\ = 5 + 6 \cos(30\pi t - 1.05\pi) + 2 \cos(80\pi t - 1.133\pi)$$

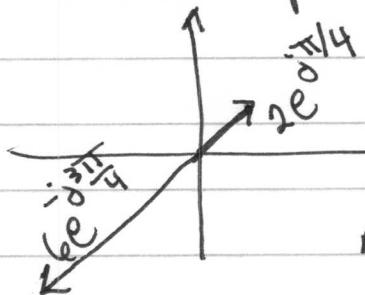
[C] $x(t) = x(t) + 2 \cos(30\pi t + 0.25\pi)$

This is adding another cosine at a frequency we already have, namely 30π , so the phasor addition theorem is used to combine the two terms at $\omega = 30\pi$.

The new complex amplitude will be

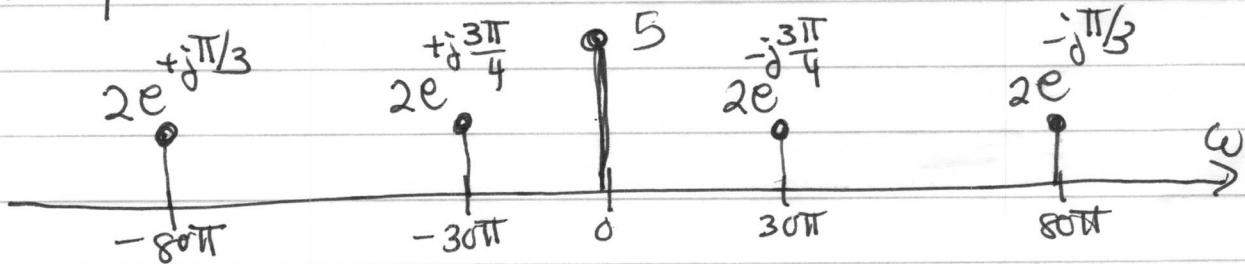
$$X_{30\pi} = 6e^{-j\frac{3\pi}{4}} + 2e^{+j\frac{\pi}{4}} = 4e^{-j\frac{3\pi}{4}}$$

You can get that using normal complex arithmetic, but it's even easier if you notice that the two complex amplitudes are 180° out of phase:



We can add this visually, and see the result is $4e^{-j\frac{3\pi}{4}}$.

Anyway, Now $x(t) = 5 + 4 \cos(30\pi t - \frac{3\pi}{4}) + 4 \cos(80\pi t - \frac{\pi}{3})$, and the spectrum is this:



3.2

To get the spectrum, we need to express each signal as a sum of cosines first.

[a] $x(t) = \sin^3(5\pi t)$. We can use the inverse Euler formula for sine, & then carry out the algebra (this is a way to derive trig identities, by the way....)

$$x(t) = \sin^3(5\pi t) = \left(\frac{1}{2j} [e^{j5\pi t} - e^{-j5\pi t}]\right)^3$$

$$= \frac{1}{8j^3} (e^{j5\pi t} - e^{-j5\pi t})^3$$

; I won't write down all the algebra for cubing this, but you just have to multiply it all out, being careful about signs. I got:

$$x(t) = \frac{1}{8j^3} \left[e^{j15\pi t} - 3e^{j5\pi t} + 3e^{-j5\pi t} - e^{-j15\pi t} \right]$$

Let's group terms of the form $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ for the same ω_0 , and also note that $j^3 = j \cdot j^2 = -j$

$$\begin{aligned} x(t) &= -\frac{1}{8j} \left[(e^{j5\pi t} - e^{-j5\pi t}) - 3(e^{j5\pi t} - e^{-j5\pi t}) \right] \\ &= +\frac{1}{4} \left(+3\sin 5\pi t - \sin 15\pi t \right) \end{aligned}$$

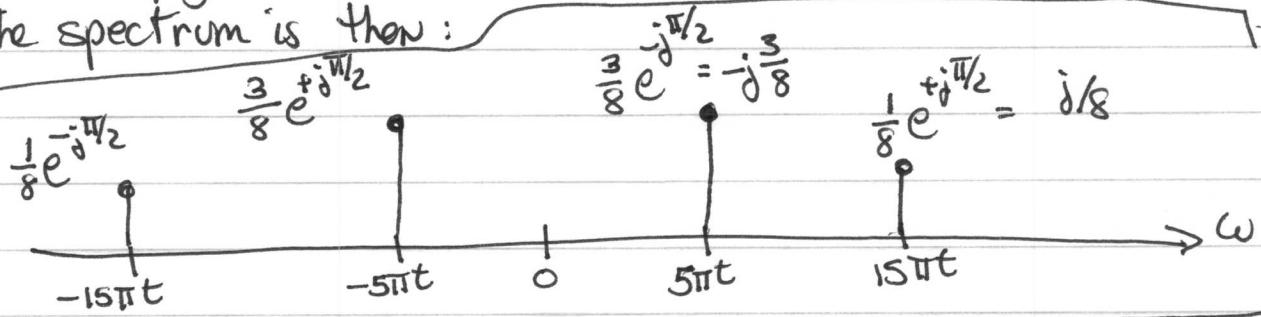
(You can verify this answer by looking up the trig identity for $\sin^3 x$). Finally, using $\pm \sin x = \cos(x \mp \pi/2)$,

$$x(t) = \frac{1}{4} (3 \cos(5\pi t - \pi/2) + \cos(15\pi t + \pi/2))$$

(5)

(The next page has a MATLAB verification of this result.)

The spectrum is then:



$$\boxed{b} \quad y(t) = [\cos(2\pi t) \cos(14\pi t)]^2 = \cos^2(2\pi t) \cdot \cos^2(14\pi t)$$

Again, we need a sum of cosines. This time, I used a trig identity for $\cos^2 x = \frac{1}{2}(1+2x)$, the inverse Euler formula and some algebra:

$$y(t) = \frac{1}{4} [(1+\cos(4\pi t)) \cdot (1+\cos(28\pi t))]$$

$$= \frac{1}{4} [1 + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})] [1 + \frac{1}{2}(e^{j28\pi t} + e^{-j28\pi t})]$$

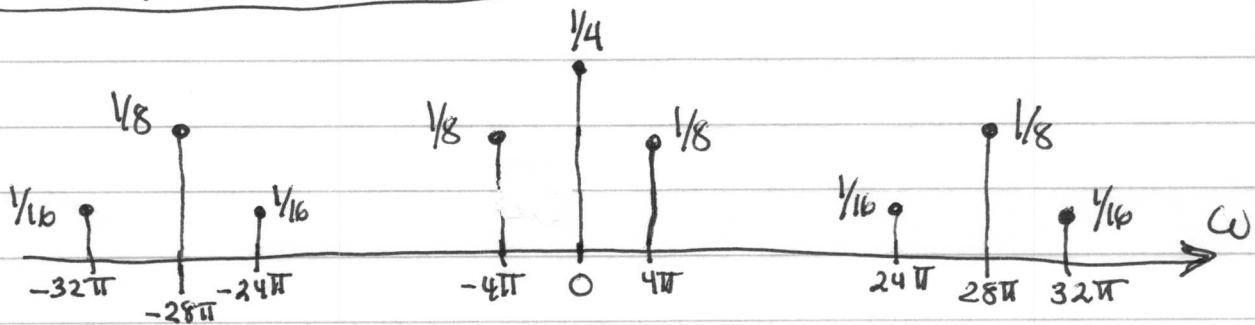
$$= \frac{1}{16} (4 + 2e^{j28\pi t} + 2e^{-j28\pi t} + 2e^{j4\pi t} + 2e^{-j4\pi t} + e^{j32\pi t} + e^{-j32\pi t} + 2e^{j24\pi t} + 2e^{-j24\pi t})$$

Grouping pairs to give me cosines, I get

$$y(t) = \frac{1}{8} [2 + 2\cos(4\pi t) + \cos(24\pi t) + 2\cos(28\pi t) + \cos(32\pi t)]$$

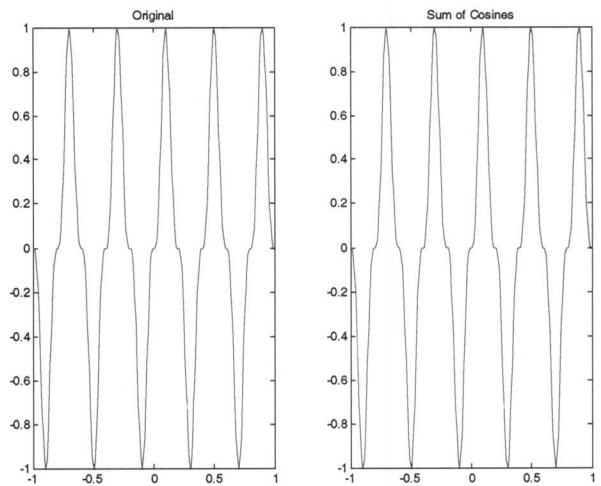
(MATLAB verification on next page)

So the spectrum is

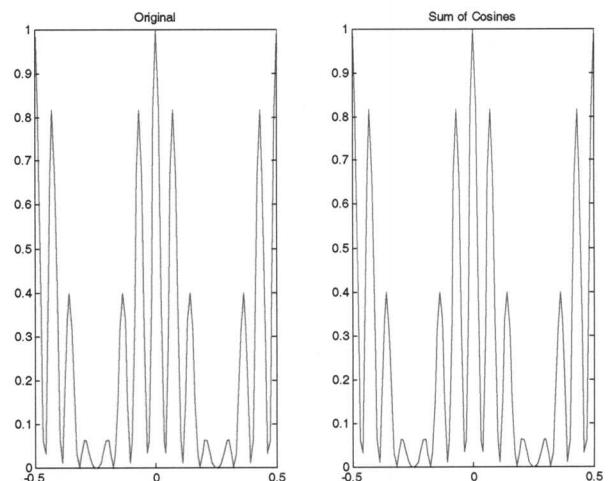


6

```
% MATLAB verification for Problem 3.2(a)
close all
clear all
t = -1:0.01:1;
% original function
x = sin(5*pi*t).^3;
figure
subplot(1,2,1);
plot(t,x)
% As sum of cosine functions
y = (3*cos(5*pi*t-pi/2) + ...
      cos(15*pi*t+pi/2))/4;
subplot(1,2,2)
plot(t,y)
```



```
% MATLAB verification for Problem 3.2(b)
t = -0.5:0.01:0.5;
% original function
x = (cos(2*pi*t).*cos(14*pi*t)).^2;
figure
subplot(1,2,1);
plot(t,x)
% As sum of cosine functions
y = (2 + 2*cos(4*pi*t)+cos(24*pi*t) + ...
      2*cos(28*pi*t) + cos(32*pi*t))/8;
subplot(1,2,2)
plot(t,y)
```



(7)

3.3

Refer to the graph (Fig. P-3.19) on p. 70 of the text.

Look at waveform (a). It appears to be a single sinusoid of ~~frequency~~ frequency about 1.2 Hz, with peak-to-null amplitude variation of 6, and centered around an average (DC) value of ± 2 . But so is waveform (c). The difference is the phase. The positive peak of (a) closest to the origin is about $1/8$ cycle to the right of $t=0$, so the cosine will have a phase of $-\pi/4$. So (a) goes with Spectrum (3), and (c) with (1).

Now look at the cases with no DC component, (b), (d), & (e). (b) has only 1 sinusoidal component while the others must have more than 1, so (b) goes with (5).

We can sort out what's left by considering fundamental frequencies and periods:

$$\text{fundamental freq } f_0 = \gcd\{f_k\} = \gcd(0.6, 1.5) = 0.3 \text{ for spectrum (2)}$$

$$= \gcd(1.2, 2) = 0.4 \text{ for #4}$$

The corresponding periods are $\frac{1}{0.3} = 3.33$ secs for the signal that goes with spectrum (2), and $\frac{1}{0.4} = 2.5$ secs for #4. So, (d) goes with # (2) & (e) goes with #4

Summary:

a	\longleftrightarrow	3
b	\longleftrightarrow	5
c	\longleftrightarrow	1
d	\longleftrightarrow	2
e	\longleftrightarrow	4

3.4

- (a) For $x(t)$ to be real, we need the spectral lines to occur in conjugate pairs, so we can combine the various $A \exp(\pm j(\omega_0 t - \phi))$ terms into real-valued cosine functions. Therefore, we need

$$\omega_1 = 40\pi, \omega_2 = 60\pi$$

$$X_{-1} = X_1^* = \sqrt{3} + j\sqrt{3}, \quad X_2 = X_{-2}^* = 2e^{-j\frac{2\pi}{3}}$$

- (b) It then follows that

$$x(t) = B + 2\sqrt{6} \cos(40\pi t - \frac{\pi}{4}) + 4 \cos(60\pi t - \frac{2\pi}{3})$$

$$(\text{this uses } \sqrt{3} - j\sqrt{3} = X_1 = \sqrt{6} e^{-j\frac{\pi}{4}})$$

- (c) The most negative value that the first cosine term can take on is $-2\sqrt{6}$; for the second, it is -4 . So the sum of the two can never be less than $8.899 = -(4+2\sqrt{6})$. (and it won't be that negative unless both cosines hit a negative "peak" at the same time).

So, $B = +8.899 = +(4+2\sqrt{6})$ will guarantee $x \geq 0$.

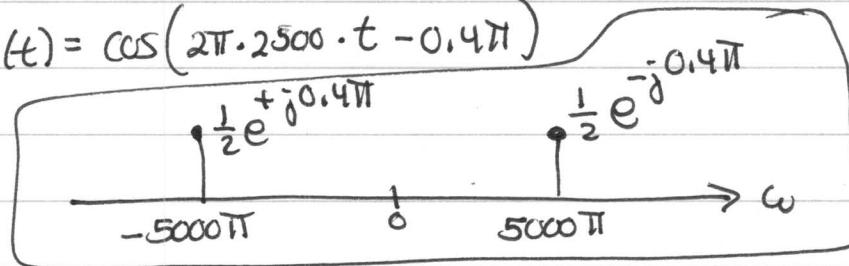
(If you plot $x(t)$ in MATLAB, you'll find that its actual minimum is about -8.6 .)

(9)

3,5

a]

$$v(t) = \cos(2\pi \cdot 2500 \cdot t - 0.4\pi)$$



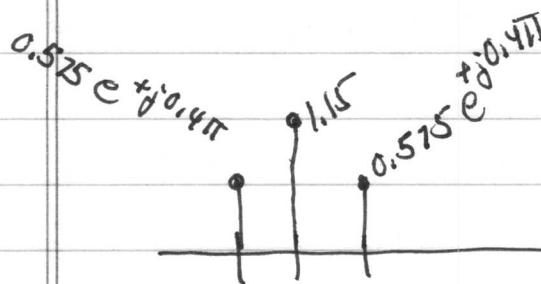
b] (To avoid writing 5000π and $1,500,000\pi$ many times, let's let $\omega_2 = 5000\pi$ & $\omega_1 = 1,500,000\pi$. I'll put in the actual numbers at the end).

$$\begin{aligned} x(t) &= [A + \cos(\omega_1 t - \varphi)] \cos \omega_2 t \\ &= A \cos \omega_2 t + A \cos(\omega_1 t - \varphi) \cos \omega_2 t \end{aligned}$$

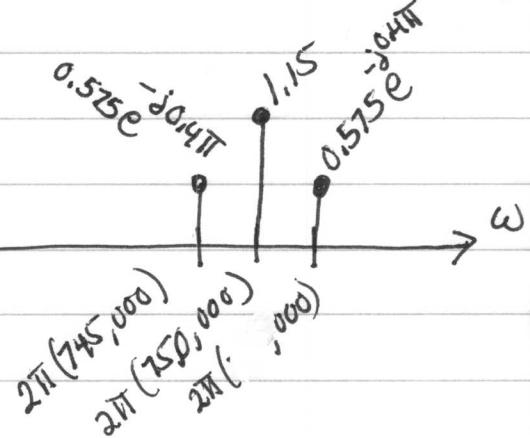
Apply the inverse Euler formula to the cosine products:

$$\begin{aligned} \cos(\omega_1 t - \varphi) \cos \omega_2 t &= \frac{1}{2} \left[e^{-j(\omega_1 t - \varphi)} + e^{j(\omega_1 t - \varphi)} \right] \cdot \frac{1}{2} \left[e^{j\omega_2 t} - e^{-j\omega_2 t} \right] \\ &= \frac{1}{4} \left[e^{j[(\omega_1 + \omega_2)t - \varphi]} + e^{-j[(\omega_1 + \omega_2)t - \varphi]} + e^{j[(\omega_1 - \omega_2)t - \varphi]} - e^{-j[(\omega_1 - \omega_2)t - \varphi]} \right] \\ &= \frac{1}{2} [\cos[(\omega_1 + \omega_2)t - \varphi] + \cos[(\omega_1 - \omega_2)t - \varphi]] \end{aligned}$$

$$\Rightarrow x(t) = 2.3 \cos(2\pi \cdot (750 \times 10^3) t) + 1.15 \left\{ \cos[2\pi(755,000)t - 0.4\pi] + \cos[2\pi(745,000)t - 0.4\pi] \right\}$$



(negatives of
the positive freqs)



3.6

Refer to the graph in the problem statement.

a

Frequency of DC component = ϕ Hz.

COSINE component: period = 1 sec = T

$$\Rightarrow f_0 = \frac{1}{T} = [1 \text{ Hz}]$$

b

The cosine has a negative peak @ $t=0$, so it is a $-\cos()$ function. But to be more formal, write it as $\cos(2\pi(1) \cdot (t - t_d))$, and use $t_d = 0.5$ because that's the first positive peak. Then we have $\cos(2\pi(1)(t - 0.5)) = \cos(2\pi t - \pi) = -\cos(2\pi t)$. Combined with the DC part, we get

$$\begin{aligned} X(t) &= 10 + 10 \cos(2\pi t - \pi) \\ &= 10 - 10 \cos(2\pi t) \end{aligned}$$

c

