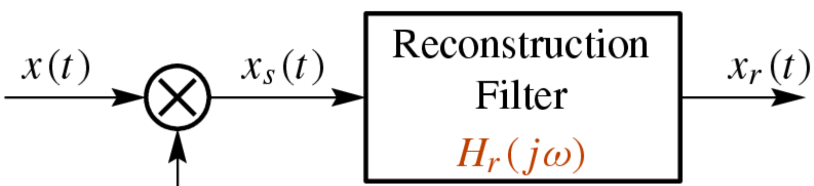
Signal Processing First

Lecture 26
Review: Digital Filtering of Analog Signals

LECTURE OBJECTIVES

- Sampling Theorem Revisited
  - GENERAL: in the FREQUENCY DOMAIN
  - Fourier transform of sampled signal
  - Reconstruction from samples
- **Effective Frequency Response**
- Important FT properties
  - Convolution ←→ multiplication
  - Frequency shifting

Sampling: Freq. Domain



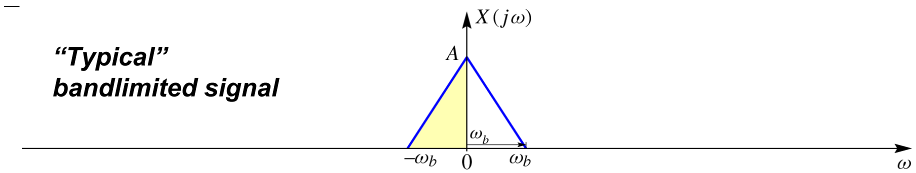
$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT FREQUENCY SHIFTING !!!

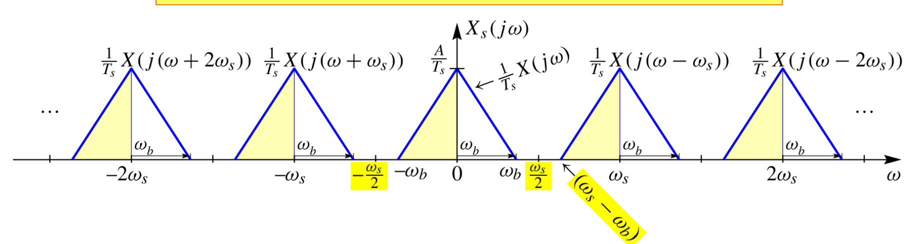
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

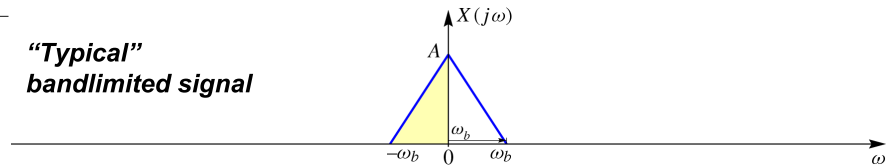


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

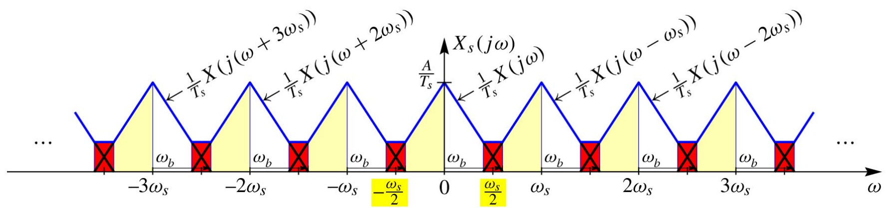


Aliasing Distortion

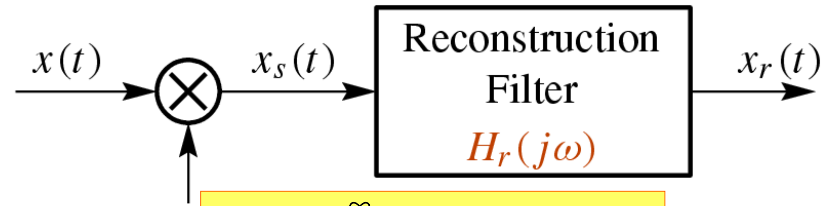
"Typical" bandlimited signal



- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Reconstruction of *x*(*t*)

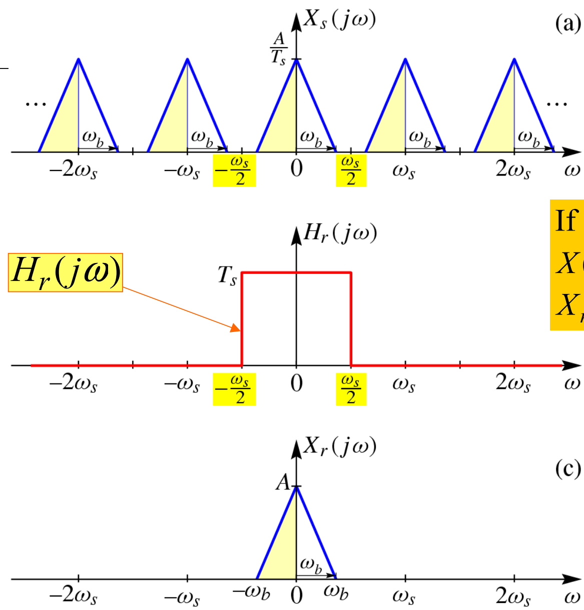


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

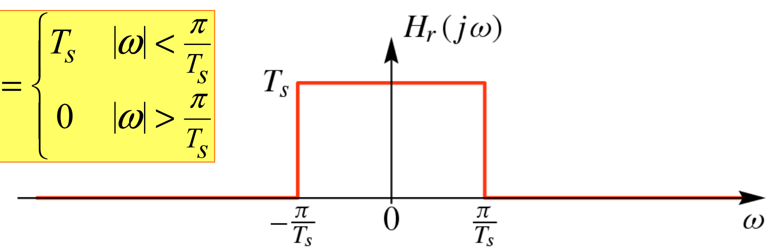
Reconstruction: Frequency-Domain



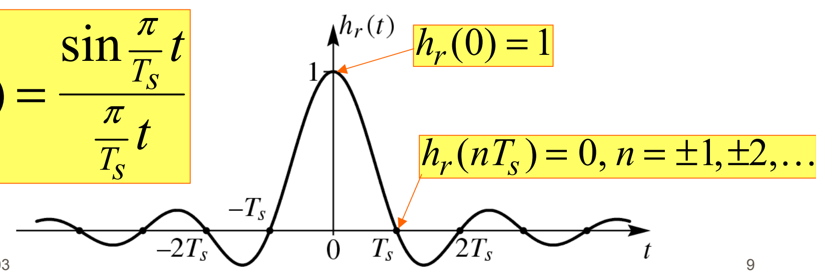
If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)X_s(j\omega)$

Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

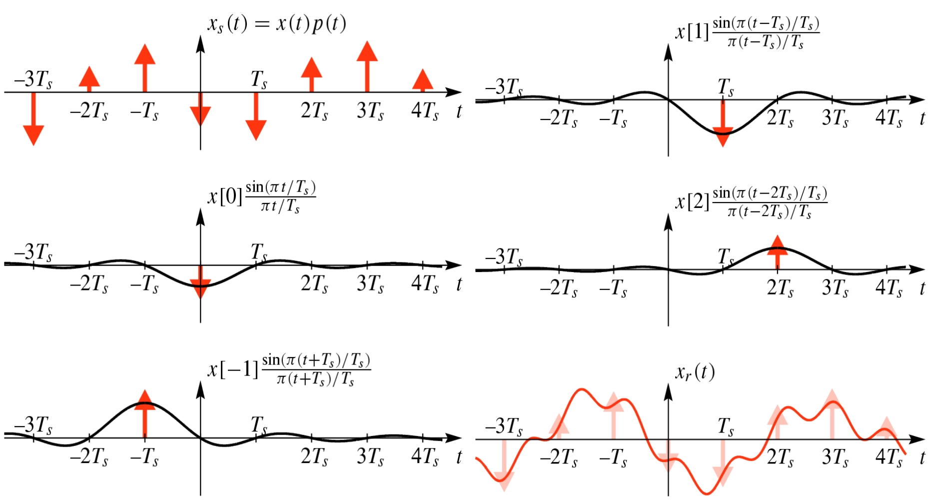
Shannon Sampling Theorem

- **"SINC" Interpolation** is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

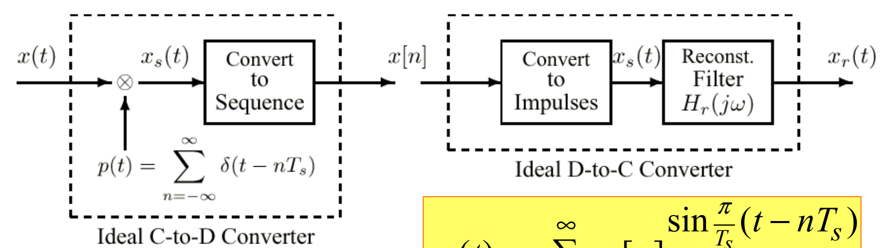
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

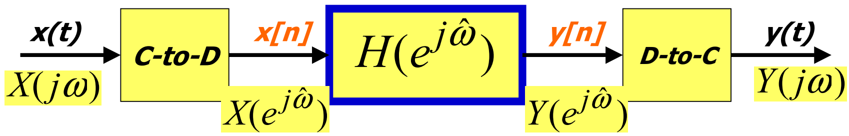
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

DT Filtering of CT Signals

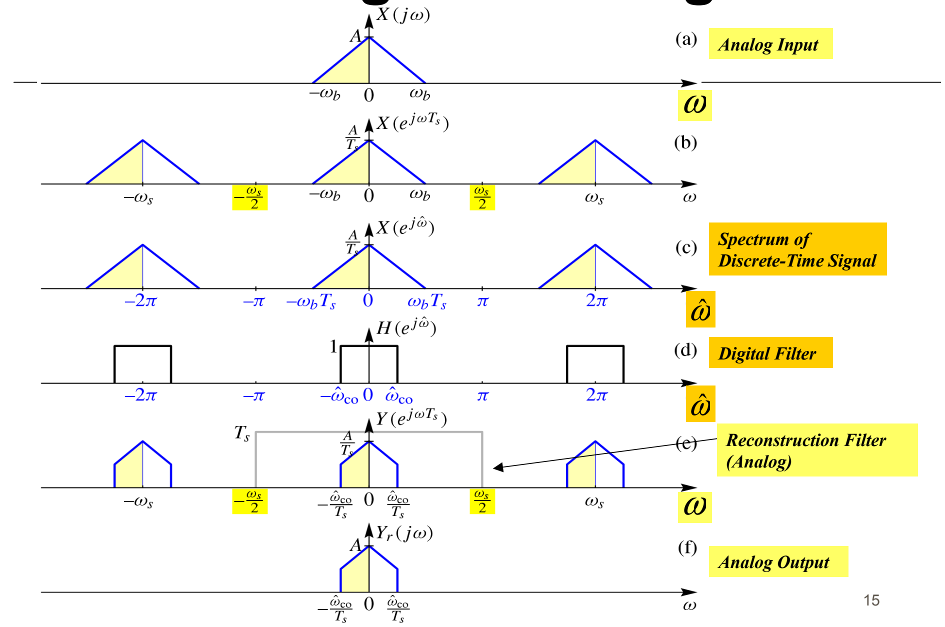


If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| < \frac{1}{2}\omega_s \\ \text{UNDEFINED NOT LTI} & |\omega| > \frac{1}{2}\omega_s \end{cases}$$

DT Filtering of a CT Signal



EFFECTIVE Freq. Response

- Assume NO Aliasing, then
  - ANALOG FREQ <--> DIGITAL FREQ

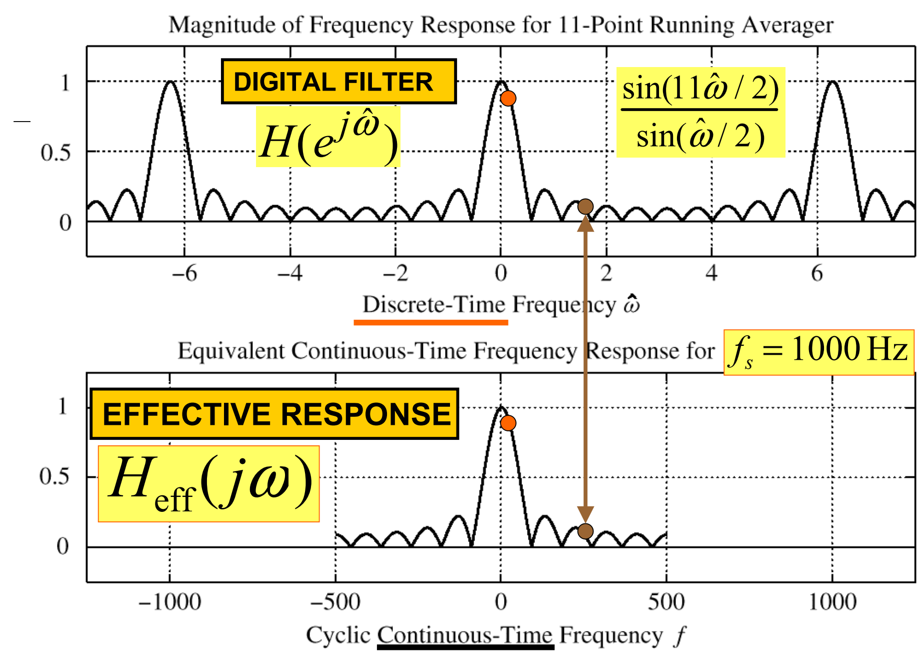
$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DIGITAL FILTER

- So, we can plot:
- Scaled Freq. Axis

$$H(e^{j\omega T_s}) \text{ vs. } \omega$$

ANALOG FREQUENCY



H_{eff} for 11-pt Averager

- Frequency Response for Discrete-time

$$H(e^{j\hat{\omega}}) = \frac{\sin(11\hat{\omega}/2)}{\sin(\hat{\omega}/2)}$$

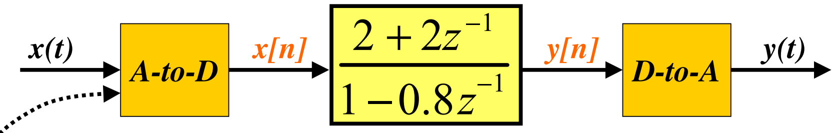
$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s} = \frac{\omega}{1000}$$

- Analog Frequency Response

$$H(j\omega) = \frac{\sin(11\omega/2000)}{\sin(\omega/2000)}$$

POP QUIZ

- Given:



- Find the output, y(t)
  - When $x(t) = \cos(2000\pi t)$

$$f_s = 5000\,\text{Hz}$$

POP QUIZ BECOMES

- Given: $H(z) = \dfrac{2+2z^{-1}}{1-0.8z^{-1}}$

- Find the output, y[n]
  - When $x[n] = \cos(0.4\pi n)$
  - Because $\omega T_s = 2000\pi / 5000 = 0.4\pi$

NO Aliasing

SINUSOIDAL RESPONSE

- x[n] = SINUSOID => y[n] is SINUSOID
- Get MAGNITUDE & PHASE from H(z)

$$\text{if } x[n] = e^{j\hat{\omega}n} \text{ then}$$

$$y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$\text{where } H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

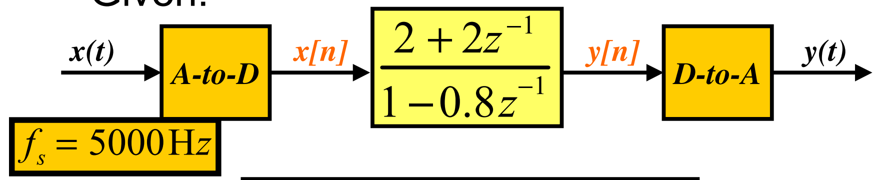
POP QUIZ INSIDE ANSWER

- Given: $H(z) = \dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- The input: $x[n] = \cos(0.4\pi n)$
- Then y[n] $y[n] = M\cos(0.4\pi n + \psi)$

$$H(e^{j0.4\pi}) = \frac{2+2e^{-j0.4\pi}}{1-0.8e^{-j0.4\pi}} = 3.02\,e^{-j0.452\pi}$$

POP QUIZ ANSWER

- Given:

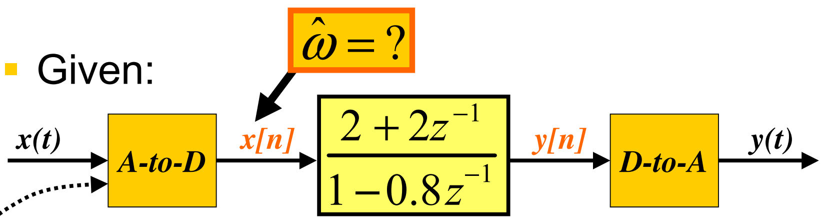


- When $x(t) = \cos(2000\pi t)$
- The output is

$$y(t) = 3.02\cos(2000\pi t - 0.452\pi)$$

ANOTHER INPUT FREQ

- Given:



- Find the output, y(t)
  - When $x(t) = \cos(2\pi(7500)t)$

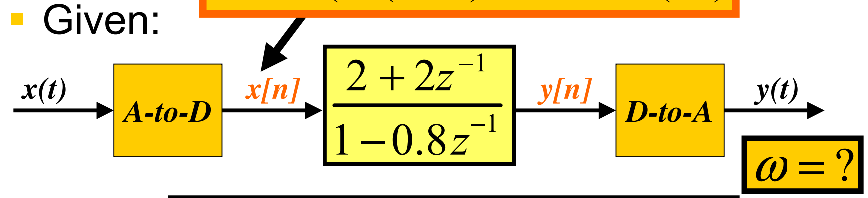
$f_s = 5000\,\text{Hz}$

$$\hat{\omega} = ?$$

2nd POP QUIZ ANSWER

- Given:

$\hat{\omega} = \cos(2\pi(7500)/5000 = 2\pi(1.5)$



- When $x(t) = \cos(2\pi(7500)t)$

$f_s = 5000\,\text{Hz}$ → $\hat{\omega} = 3\pi$ → $y(t) = ?$

IMPORTANT CONCEPTS

- ALL Signals have Frequency Content
  - Sum of Sinusoids
  - Complex Exponentials
  - Impulses, Square Pulses
- FILTERS alter the Frequency Content
  - Image Processing Example: Blur
  - Linear Time-Invariant Processing
- 3 Domains for Analysis

Superficial Knowledge

- It depends how carefully you think about it. If you don't think very carefully it's obvious; but if you think about it in depth, you'll get confused and it won't be obvious.

…anon

THREE DOMAINS

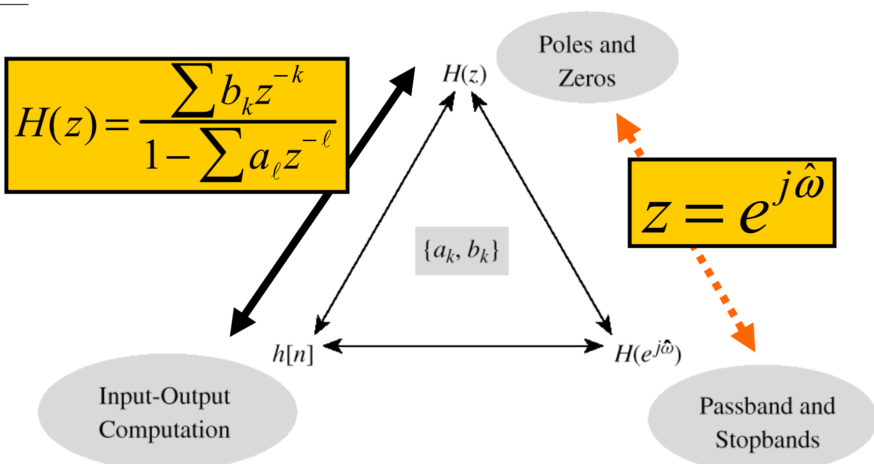
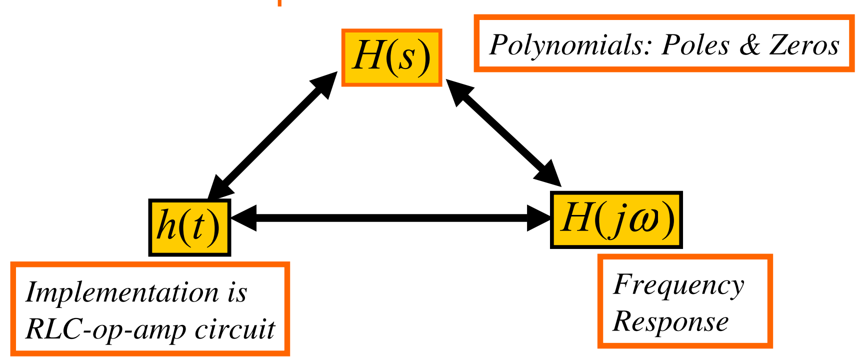


Figure 8.13 Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k,\ b_k\}$ play a central role.

THE FUTURE

- Circuits & Laplace Transforms



Mathematical Elegance

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier
(Inverse

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis
(Forward Transform)

$$\text{Time - domain} \Leftrightarrow \text{Frequency - domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$