Signal Processing First

Lecture 20
Convolution
(Continuous-Time)

READING ASSIGNMENTS

- This Lecture:
  - Chapter 9, Sects. 9-6, 9-7, and 9-8
- Other Reading:
  - Recitation: Ch. 9, all
  - Next Lecture: Start reading Chapter 10

LECTURE OBJECTIVES

- Review of C-T LTI systems
- **Evaluating convolutions**
  - **Examples**
  - **Impulses**
- LTI Systems
  - Stability and causality
  - Cascade and parallel connections

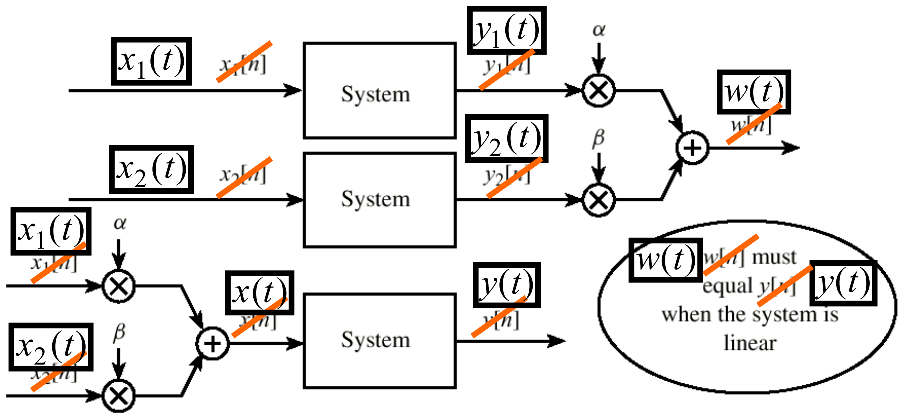
Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output *y(t)* is related to the input *x(t)* by a ***convolution integral***

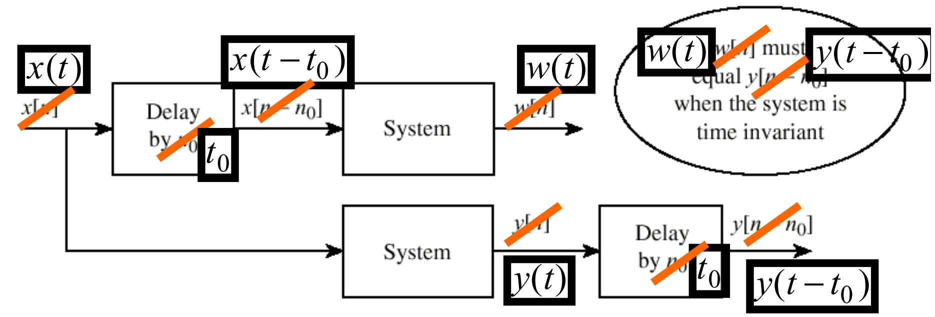
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

where *h(t)* is the ***impulse response*** of the system.

Testing for Linearity



Testing Time-Invariance



Ideal Delay: $y(t) = x(t - t_d)$

- Linear

$$a x_1(t - t_d) + b x_2(t - t_d) = a y_1(t) + b y_2(t)$$

- and Time-Invariant

$$w(t) = x((t - t_d) - t_0)$$

$$y(t - t_0) = x((t - t_0) - t_d)$$

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

- Linear

$$\int_{-\infty}^{t} [a x_1(\tau) + b x_2(\tau)]\,d\tau = \int_{-\infty}^{t} a x_1(\tau)\,d\tau + \int_{-\infty}^{t} b x_2(\tau)\,d\tau$$
$$= a y_1(t) + b y_2(t)$$

- And Time-Invariant

$$w(t) = \int_{-\infty}^{t} x(\tau - t_0)\,d\tau \qquad \text{Let } \sigma = \tau - t_0$$

$$\Rightarrow \quad w(t) = \int_{-\infty}^{t-t_0} x(\sigma)\,d\sigma = y(t - t_0)$$

Modulator: $y(t) = [A + x(t)]\cos\omega_c t$

- ***Not*** linear--obvious because

$$[A + ax_1(t) + bx_2(t)] \neq$$
$$[A + ax_1(t)] + [A + bx_2(t)]$$

- ***Not*** time-invariant

$$w(t) = [A + x(t - t_0)]\cos\omega_c t \neq y(t - t_0)$$

Linear and Time-Invariant (LTI) Systems

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where *h(t)* is the ***impulse response*** of the system.

Convolution of Impulses, etc.

- Convolution of two impulses

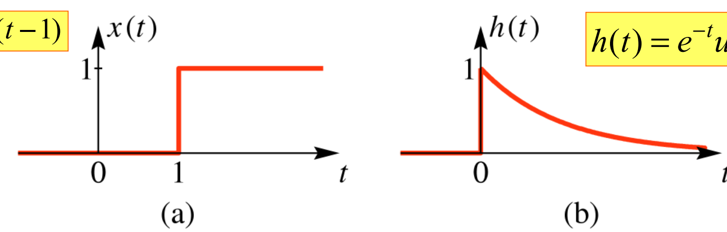
$$\delta(t - t_1) * \delta(t - t_2) = \delta(t - t_1 - t_2)$$

- Convolution of step and shifted impulse

$$u(t) * \delta(t - t_0) = u(t - t_0)$$

Evaluating a Convolution

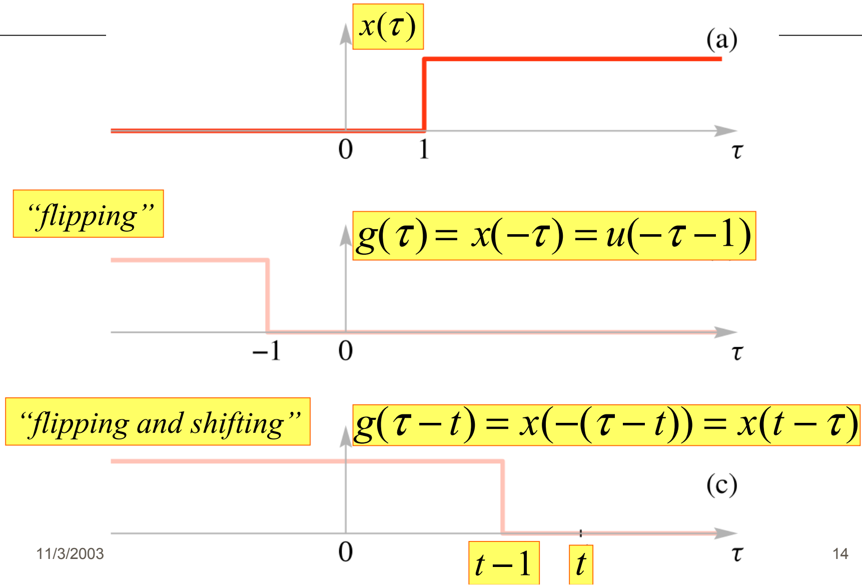
$x(t) = u(t-1)$



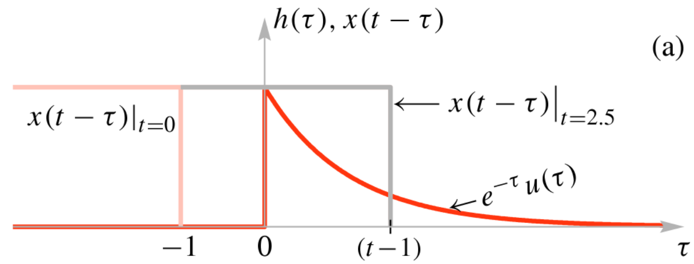
$h(t) = e^{-t}u(t)$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$$

"Flipping and Shifting"



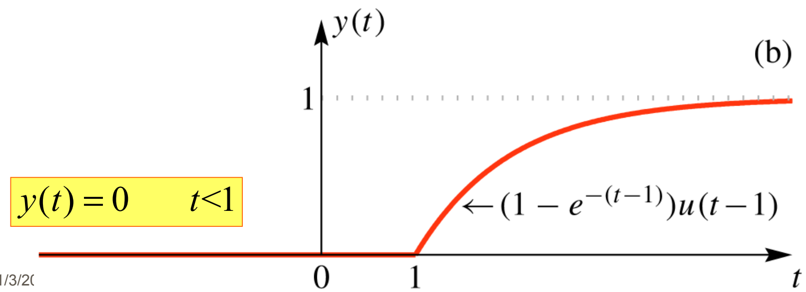
Evaluating the Integral



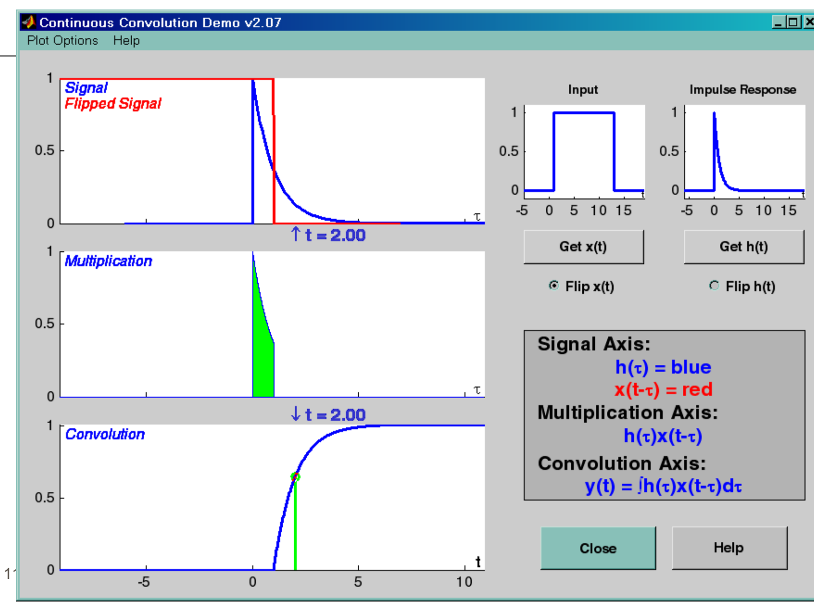
$$y(t) = \begin{cases} 0 & t-1<0 \\ \int_0^{t-1} e^{-\tau} d\tau & t-1>0 \end{cases}$$

Solution

$$\begin{aligned} y(t) &= \int_0^{t-1} e^{-\tau} d\tau = -e^{-\tau}\Big|_0^{t-1} \\ &= 1-e^{-(t-1)} \qquad t \geq 1 \end{aligned}$$



Convolution GUI



General Convolution Example

$x(t) = e^{-at}u(t)$ $\quad$ $h(t) = e^{-bt}u(t), \quad b \neq a$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau = \begin{cases} e^{-bt}\int_0^t e^{-a\tau}e^{b\tau}d\tau & t>0 \\ 0 & t<0 \end{cases}$$

$$= \begin{cases} \dfrac{e^{-at}-e^{-bt}}{-a+b} & t>0 \\ 0 & t<0 \end{cases} = \frac{e^{-at}-e^{-bt}}{b-a}u(t)$$

Special Case: u(t)

$x(t) = e^{-at}u(t), \quad a \neq 0$ $\quad$ $h(t) = u(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

$$= \frac{1}{a}(1-e^{-at})u(t)$$

if $a = 2$

$$y(t) = \frac{1}{2}(1-e^{-2t})u(t)$$

Convolve Unit Steps

$x(t) = u(t)$ $\quad$ $h(t) = u(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

$$= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t 1\,d\tau & t>0 \\ 0 & t<0 \end{cases}$$

$$= \begin{cases} t & t>0 \\ 0 & t<0 \end{cases} = tu(t)$$

Unit Ramp

Convolution is Commutative

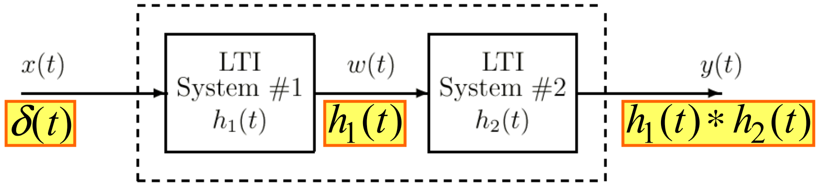
$$h(t)*x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

let $\sigma = t-\tau$ and $d\sigma = -d\tau$

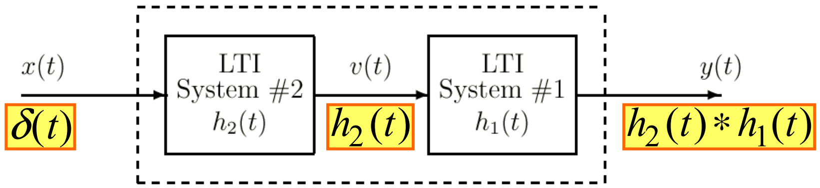
$$h(t)*x(t) = -\int_{\infty}^{-\infty} h(t-\sigma)x(\sigma)d\sigma$$

$$= \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma = x(t)*h(t)$$

Cascade of LTI Systems



$$h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t)$$



(b)

Stability

- A system is stable if every bounded input produces a bounded output.
- A continuous-time *LTI system* is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Causal Systems

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \le t_0$.
- An LTI system is causal if and only if

$$h(t) = 0 \ \text{ for } \ t < 0$$

Convolution is Linear

- Substitute $x(t)=ax_1(t)+bx_2(t)$

$$y(t) = \int_{-\infty}^{\infty} [ax_1(\tau) + bx_2(\tau)]h(t-\tau)d\tau$$

$$= a\int_{-\infty}^{\infty} x_1(\tau)h(t-\tau)d\tau + b\int_{-\infty}^{\infty} x_2(\tau)h(t-\tau)d\tau$$

$$= ay_1(t) + by_2(t)$$

Therefore, convolution is linear.

Convolution is Time-Invariant

- Substitute $x(t-t_0)$

$$w(t) = \int_{-\infty}^{\infty} h(\tau)x((t-\tau)-t_o)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x((t-t_o)-\tau)d\tau$$

$$= y(t-t_o)$$