Signal Processing First

Lecture 16
IIR Filters: Feedback and H(z)

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sects. 8-1, 8-2 & 8-3
- Other Reading:
  - Recitation: Ch. 8, Sects 8-1 thru 8-4
    - POLES & ZEROS
  - Next Lecture: Chapter 8, Sects. 8-4 8-5 & 8-6

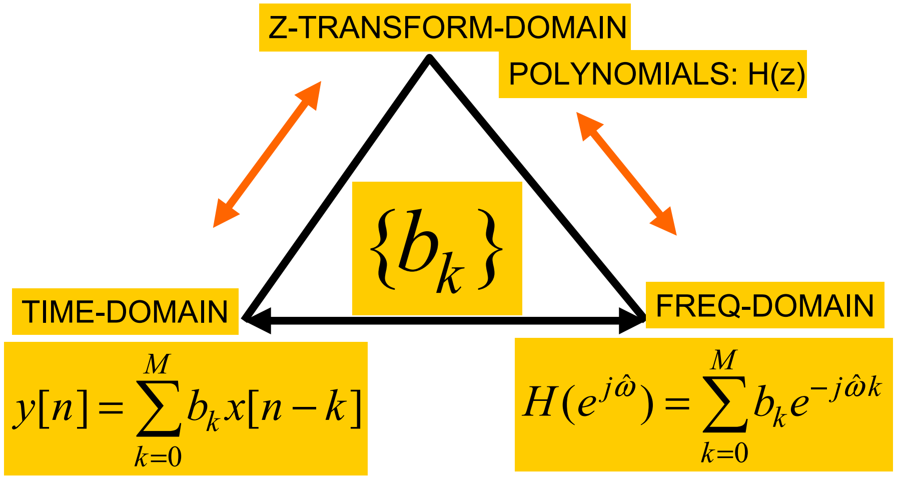
LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define **IIR** DIGITAL Filters
  - Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n]=\sum_{\ell=1}^{N}a_\ell y[n-\ell]+\sum_{k=0}^{M}b_k x[n-k]$$

- Show how to compute the output y[n]
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response h[n] ←→ H(z)

THREE DOMAINS



$$y[n]=\sum_{k=0}^{M}b_k x[n-k]$$

$$H(e^{j\hat{\omega}})=\sum_{k=0}^{M}b_k e^{-j\hat{\omega}k}$$

Quick Review: Delay by n_d

$$y[n] = x[n - n_d]$$

IMPULSE RESPONSE $h[n] = \delta[n - n_d]$

SYSTEM FUNCTION $H(z) = z^{-n_d}$

FREQUENCY RESPONSE $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega} n_d}$

LOGICAL THREAD

- FIND the IMPULSE RESPONSE, h[n]
  - INFINITELY LONG
  - **<u>IIR</u>** Filters

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS

FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

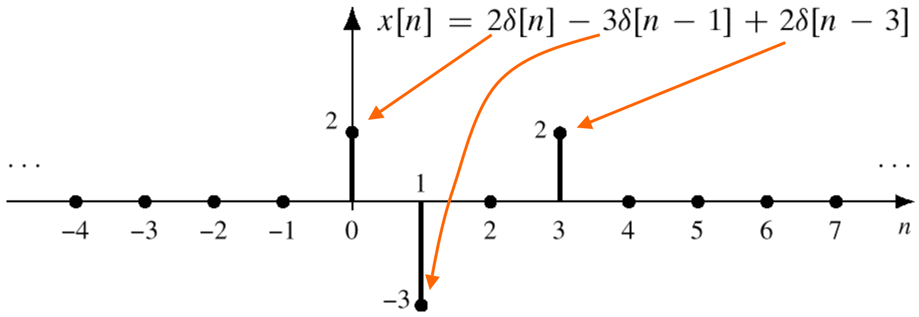
FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$



$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$

COMPUTE y[n]

- FEEDBACK DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$

- NEED y[-1] to get started

$$\boxed{y[0] = 0.8y[-1] + 5x[0]}$$

AT REST CONDITION

- y[n] = 0, for n<0
- BECAUSE x[n] = 0, for n<0

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$\boxed{y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]}$$

COMPUTE MORE y[n]

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

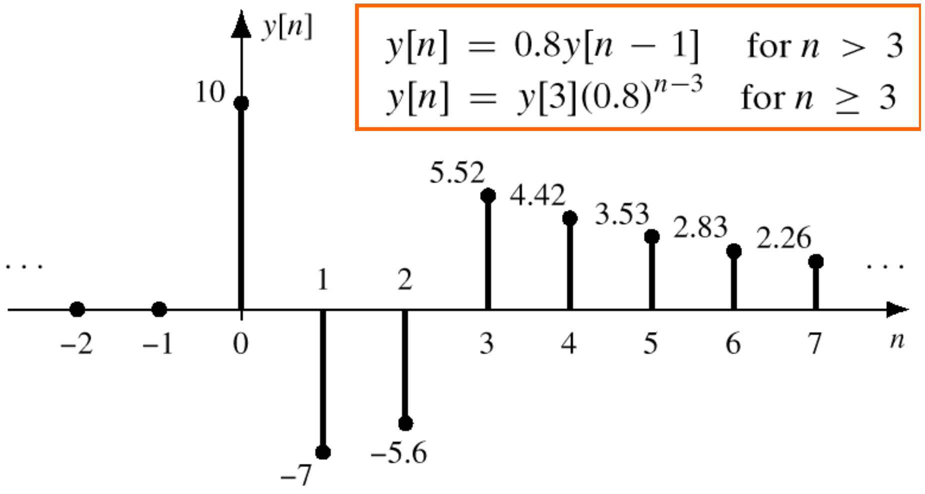
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT y[n]



$$\begin{aligned} y[n] &= 0.8y[n-1] && \text{for } n > 3 \\ y[n] &= y[3](0.8)^{n-3} && \text{for } n \geq 3 \end{aligned}$$

IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0(a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

IMPULSE RESPONSE

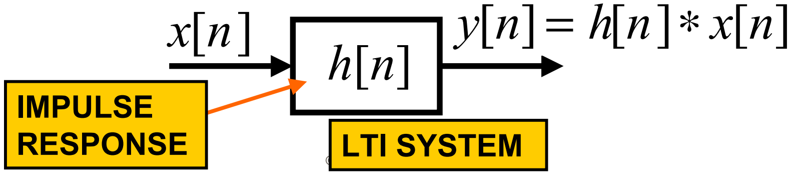
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n]$$

- Find h[n]

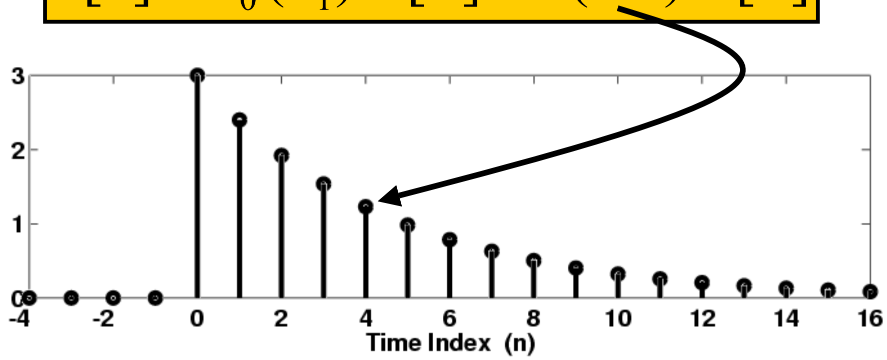
$$h[n] = 3(0.8)^n u[n]$$

- **CONVOLUTION** in TIME-DOMAIN



PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite-Length Signal: h[n]

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 (a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of H(z)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

H(z) = z-Transform{ h[n] }

- ANOTHER FIRST-ORDER IIR FILTER:

$$\boxed{y[n]=a_1y[n-1]+b_0x[n]+b_1x[n-1]}$$

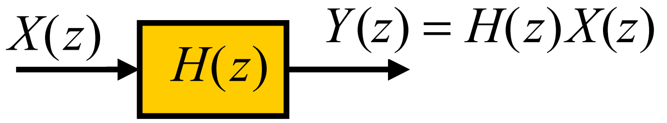
$$\boxed{h[n]=b_0(a_1)^n u[n]+b_1(a_1)^{n-1}u[n-1]}$$

z^{-1} is a shift

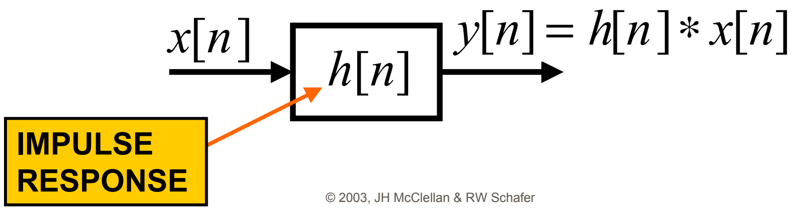
$$\boxed{H(z)=\frac{b_0}{1-a_1z^{-1}}+\frac{b_1z^{-1}}{1-a_1z^{-1}}=\frac{b_0+b_1z^{-1}}{1-a_1z^{-1}}}$$

CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



STEP RESPONSE: x[n]=u[n]

$$y[n]=a_1y[n-1]+b_0x[n]$$

$$\boxed{u[n]=1,\quad \text{for } n\ge 0}$$

n	$x[n]$	$y[n]$
$n<0$	0	0
0	1	b_0
1	1	$b_0+b_0(a_1)$
2	1	$b_0+b_0(a_1)+b_0(a_1)^2$
3	1	$b_0(1+a_1+a_1^2+a_1^3)$
4	1	$b_0(1+a_1+a_1^2+a_1^3+a_1^4)$
.		.

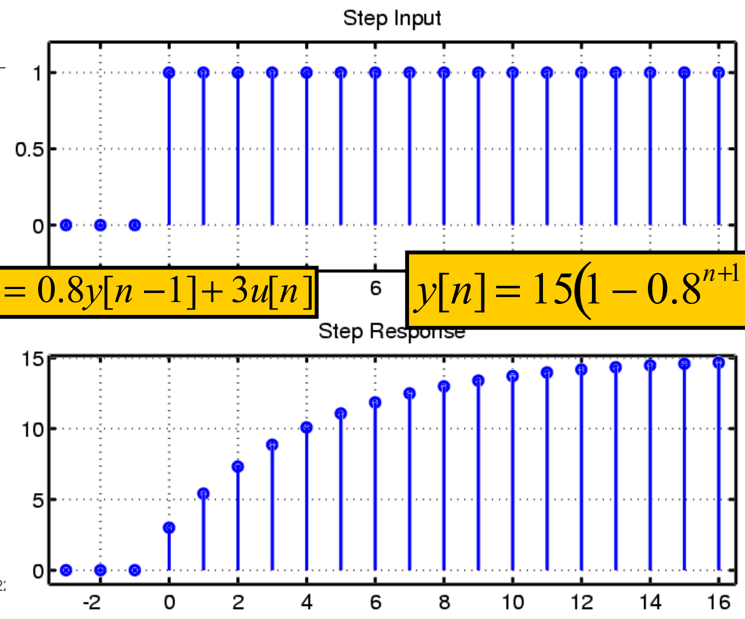
DERIVE STEP RESPONSE

$$y[n]=b_0(1+a_1+a_1^2+\ldots+a_1^n)=b_0\sum_{k=0}^{n}a_1^k$$

$$\sum_{k=0}^{L}r^k=\begin{cases}\dfrac{1-r^{L+1}}{1-r} & r\neq 1\\ L+1 & r=1\end{cases}$$

$$y[n]=b_0\frac{1-a_1^{n+1}}{1-a_1}\qquad \text{for } n\ge 0,\qquad \text{if } a_1\neq 1$$

PLOT STEP RESPONSE



$y[n] = 0.8y[n-1] + 3u[n]$

$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$