Signal Processing First

Lecture 7

Fourier Series & Spectrum

READING ASSIGNMENTS

- This Lecture:
  - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
    - Replaces pp. 62-66 in Ch 3 in DSP First
    - Notation: a_k for Fourier Series
- Other Reading:
  - Next Lecture: Sampling

LECTURE OBJECTIVES

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\mathbf{x(t+T_0) = x(t)}$

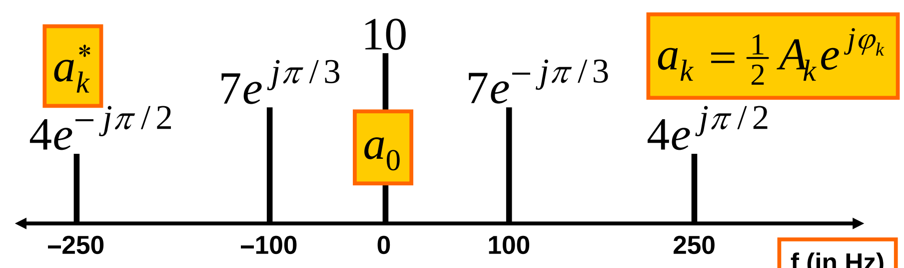
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **SPECTRUM** from Fourier Series
  - a_k is Complex Amplitude for k-th Harmonic

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq

$$a_k = \tfrac{1}{2}A_k e^{j\varphi_k}$$



$$x(t) = a_0 + \sum_{k=1}^{N}\left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t}\right\}$$

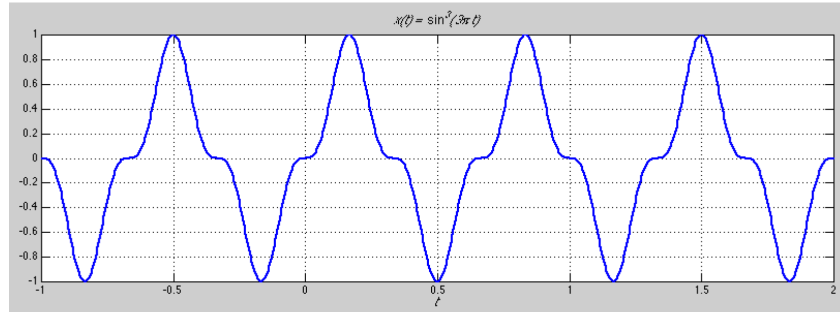
Harmonic Signal

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\boxed{2\pi\left(f_0\right) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}}$$

Example $x(t) = \sin^3(3\pi t)$

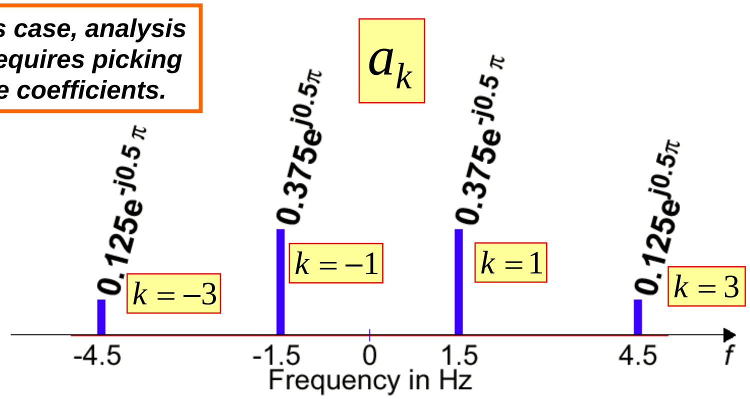


$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

Example $x(t) = \sin^3(3\pi t)$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



STRATEGY: x(t) → a_k

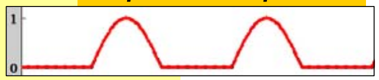
- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}\,dt$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \tfrac{1}{T_0}\int_0^{T_0/2} \sin(\tfrac{2\pi}{T_0}t)\,e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t}dt - \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t}dt$$

$$= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

$$= \tfrac{1}{4\pi(k-1)}\left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1\right) - \tfrac{1}{4\pi(k+1)}\left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1\right)$$

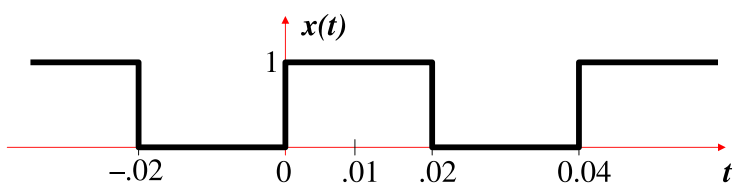
$$= \tfrac{1}{4\pi(k-1)}\left(e^{-j\pi(k-1)} - 1\right) - \tfrac{1}{4\pi(k+1)}\left(e^{-j\pi(k+1)} - 1\right)$$

$$= \left(\tfrac{k+1-(k-1)}{4\pi(k^2-1)}\right)\left(-(-1)^k - 1\right) = \begin{cases} 0 & k \text{ odd} \\ \pm\tfrac{1}{j4} & k = \pm 1 \\ \tfrac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases}$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



Fourier Coefficients a_k

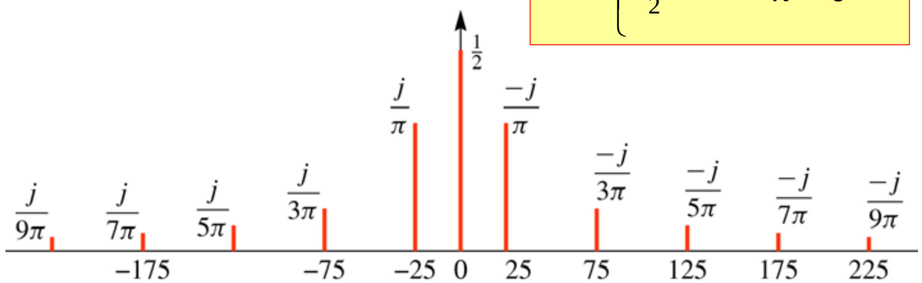
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \tfrac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$\omega_0 = 2\pi/(0.04) = 2\pi(25)$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Synthesis

- HOW do you **APPROXIMATE** x(t) ?

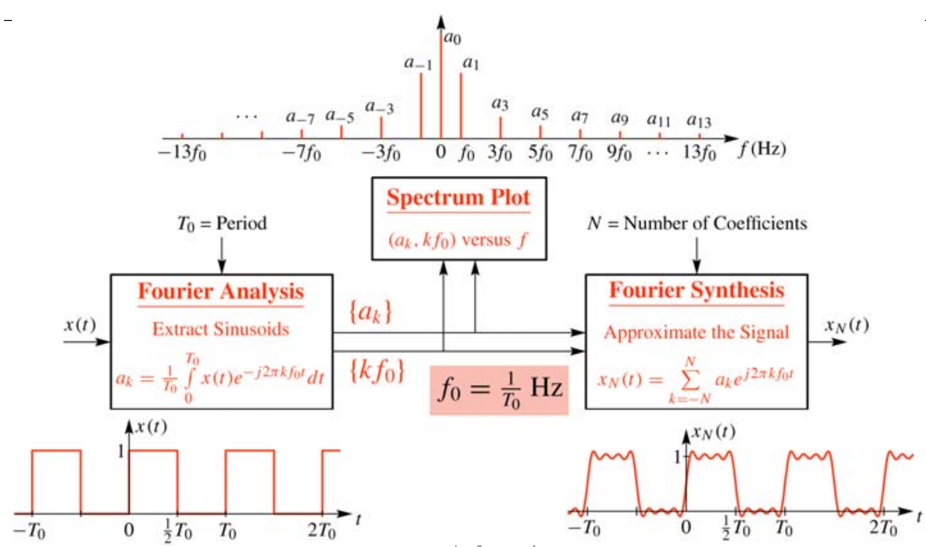
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

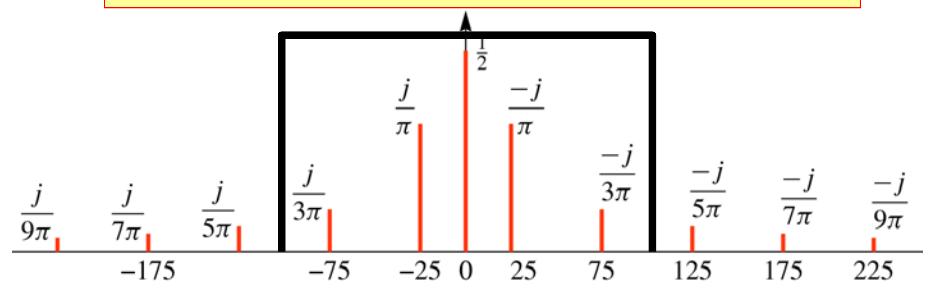
$a_{-k} = a_k^*$ when $x(t)$ is real

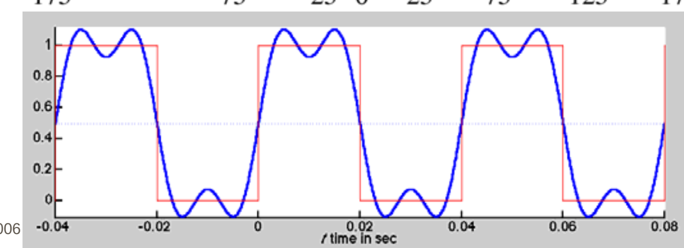
Fourier Series Synthesis



Synthesis: 1st & 3rd Harmonics

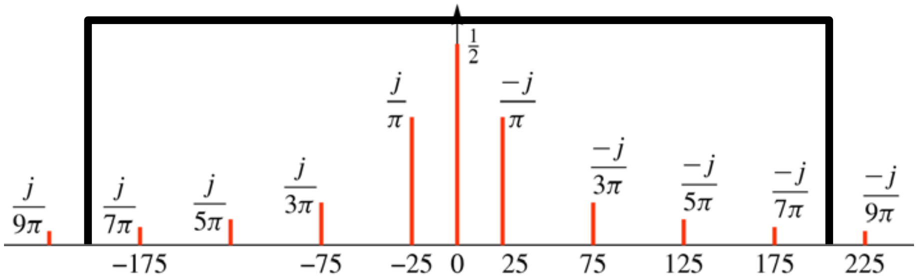
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

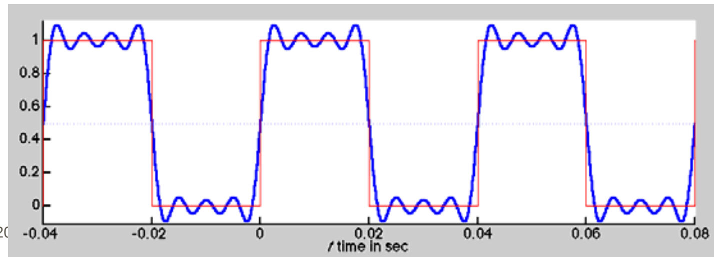




Synthesis: up to 7th Harmonic

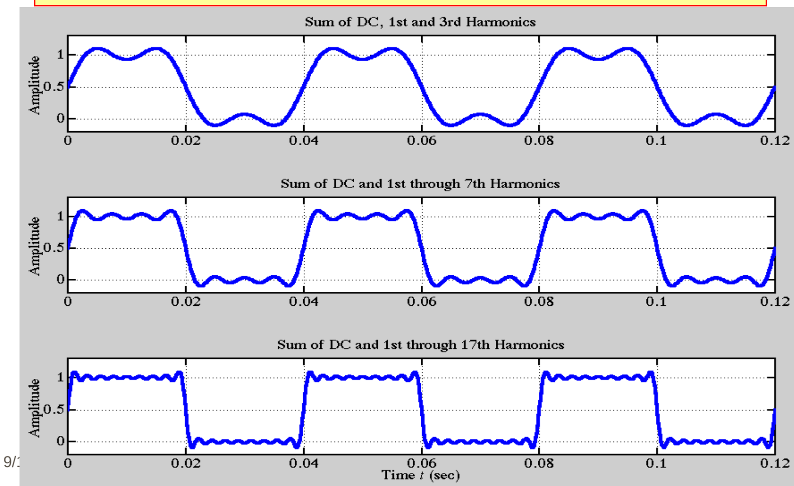
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$





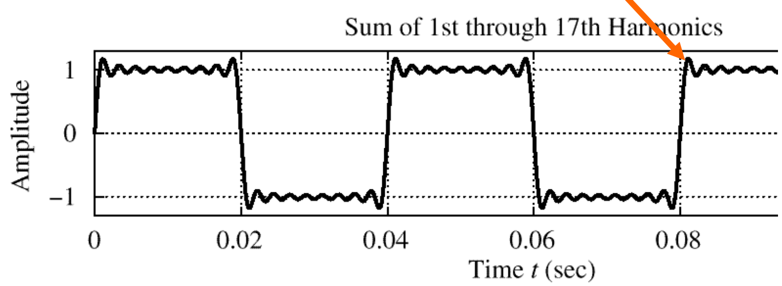
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case



Fourier Series Demos

- Fourier Series Java Applet
  - Greg Slabaugh
    - Interactive
  - http://users.ece.gatech.edu/mcclella/2025/Fsdemo_Slabaugh/fourier.html
- MATLAB GUI: fseriesdemo
  - http://users.ece.gatech.edu/mcclella/matlabGUIs/index.html

fseriesdemo GUI