Signal Processing First

Lecture 6
Fourier Series Coefficients

READING ASSIGNMENTS

- This Lecture:
  - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
    - Replaces pp. 62-66 in Ch 3 in DSP First
    - Notation: a_k for Fourier Series
- Other Reading:
  - Next Lecture: More Fourier Series

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $x(t+T_0) = x(t)$
  - Later: spectrum from the Fourier Series

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

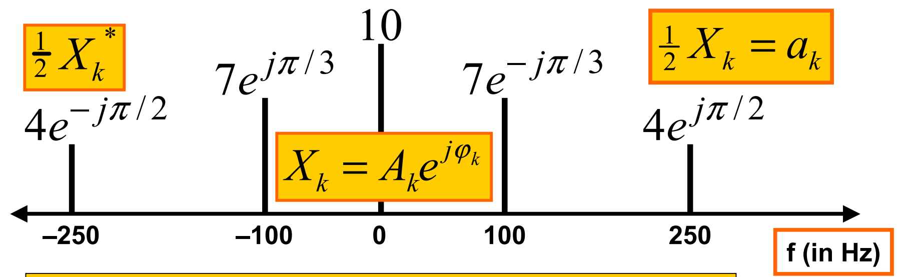
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = \boxed{a_0} + \sum_{k=1}^{N} \left\{ \boxed{a_k}\, e^{j2\pi f_k t} + \boxed{a_k^*}\, e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

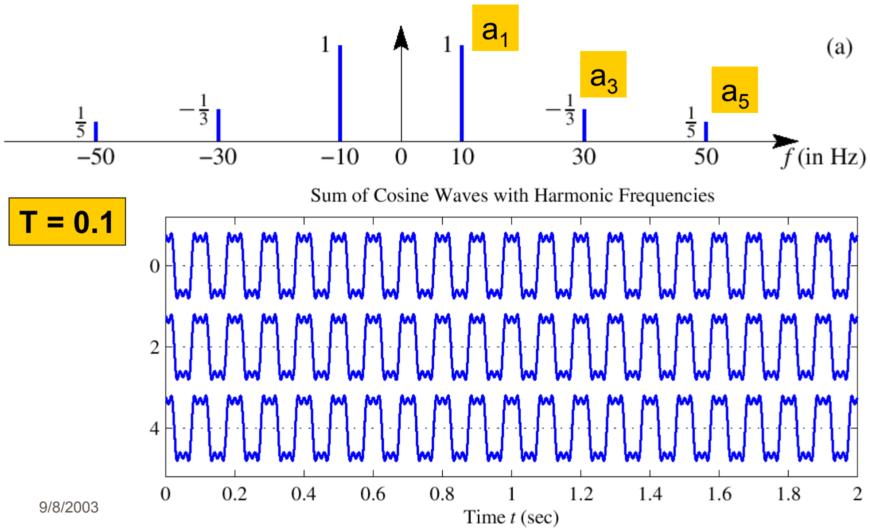
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given (ω_k,A_k,ϕ_k) create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - **Hard**
  - Given x(t), extract (ω_k,A_k,ϕ_k)
  - How many?
  - Need algorithm for computer

STRATEGY: x(t) → a_k

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt$$

INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = \frac{T_0}{-j2\pi m}e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}(e^{-j2\pi m}-1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = 0$$

$$m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- PRODUCT of exp(+j) and exp(-j)

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt}\,dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t}\,dt$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}\,dt = \frac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}\right)e^{-j(2\pi/T_0)\ell t}\,dt$$

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}\,dt = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt} e^{-j(2\pi/T_0)\ell t}\,dt\right) = a_\ell$$

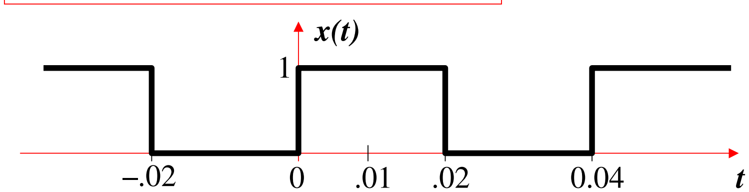
$$\Rightarrow a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt$$

Integral is zero except for $k = \ell$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1\,e^{-j(2\pi/.04)kt}\,dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k=0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1\,dt = \frac{1}{.04}(.02-0) = \tfrac{1}{2}$$

Fourier Coefficients a_k

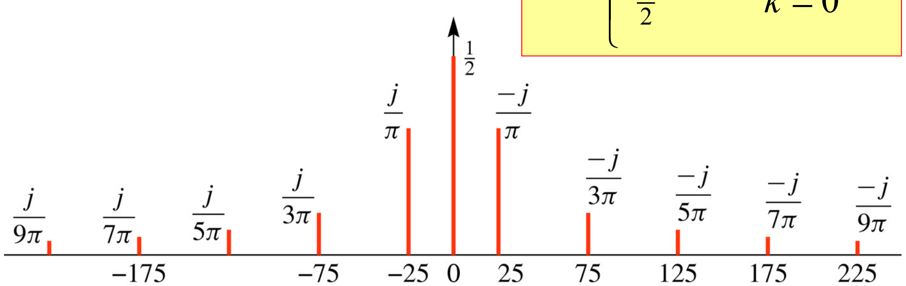
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k=\pm1,\pm3,\ldots \\ 0 & k=\pm2,\pm4,\ldots \\ \tfrac{1}{2} & k=0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k=\pm1,\pm3,\ldots \\ 0 & k=\pm2,\pm4,\ldots \\ \tfrac{1}{2} & k=0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \tfrac{1}{T_0}\int_0^{T_0} x(t)dt \qquad \text{(DC component)}$$