Signal Processing First

LECTURE #3
Phasor Addition Theorem

READING ASSIGNMENTS

- This Lecture:
  - Chapter 2, Section 2-6
- Other Reading:
  - Appendix A: Complex Numbers
  - Appendix B: MATLAB
  - Next Lecture: start Chapter 3

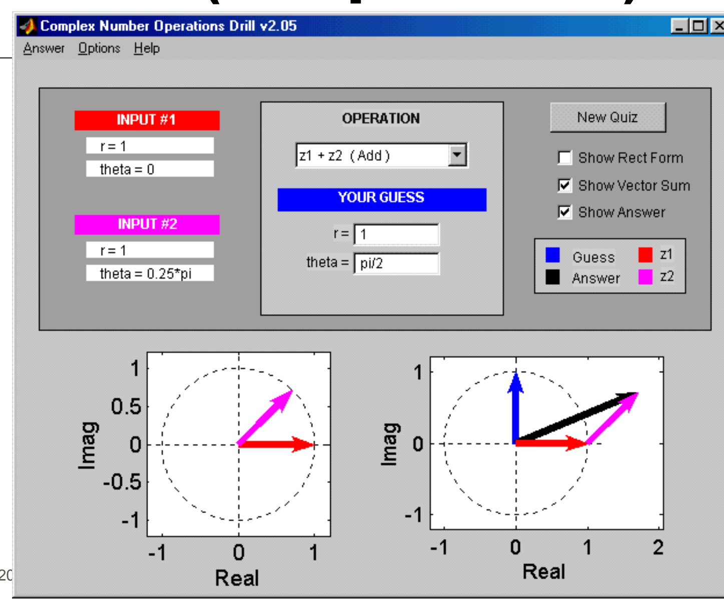
LECTURE OBJECTIVES

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids

$$\boxed{z(t) = Xe^{j\omega t} = (Ae^{j\varphi})e^{j\omega t}}$$

- Develop the ABSTRACTION:
  - Adding Sinusoids = Complex Addition
  - **PHASOR ADDITION THEOREM**

Z DRILL (Complex Arith)



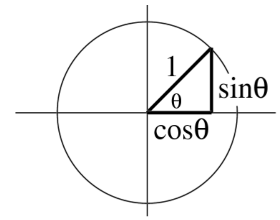
AVOID Trigonometry

- Algebra, even complex, is EASIER !!!
- Can you recall $\cos(\theta_1+\theta_2)$?
- Use: real part of $e^{j(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2)$

$$e^{j(\theta_1+\theta_2)} = e^{j\theta_1}e^{j\theta_2}$$
$$= (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$$
$$= \boxed{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)} + j(\ldots)$$

Euler's FORMULA

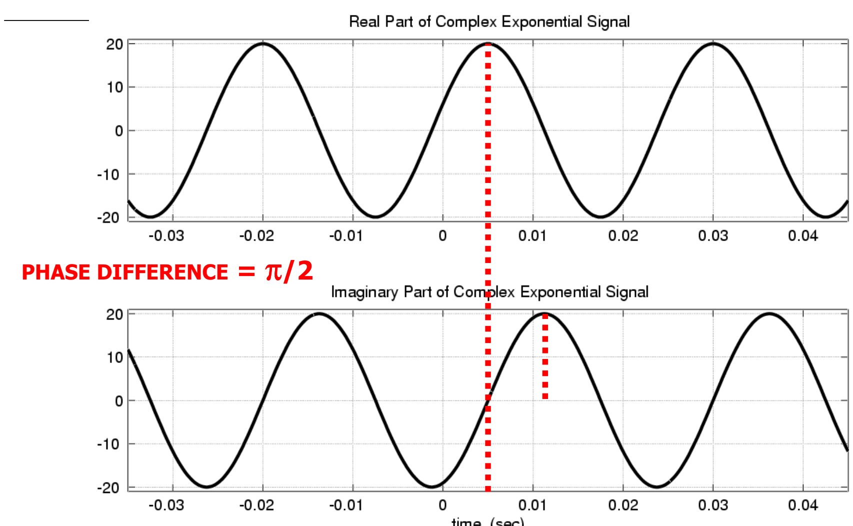
- Complex Exponential
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

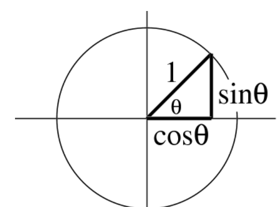
Real & Imaginary Part Plots



COMPLEX EXPONENTIAL

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

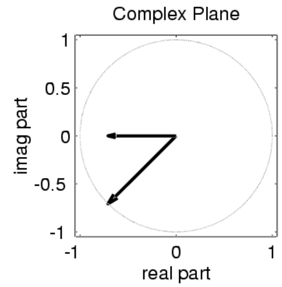
- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega=20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs

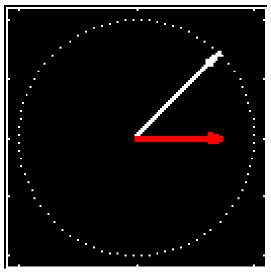


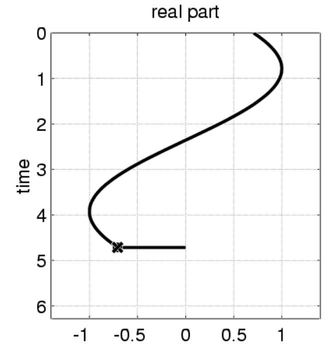
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Rotating Phasor

See Demo on CD-ROM
Chapter 2







Cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re e\left\{e^{j\omega t}\right\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j(\omega t + \varphi)}\right\}$$

$$= \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Sinusoid = REAL PART of $(Ae^{j\phi})e^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\{z(t)\}$$

Complex AMPLITUDE = X

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

POP QUIZ: Complex Amp

- Find the COMPLEX AMPLITUDE for:

$$\boxed{x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)}$$

- Use EULER's FORMULA:

$$x(t) = \Re e\left\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\right\}$$

$$= \Re e\left\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\right\}$$

$$\boxed{X = \sqrt{3}e^{j0.5\pi}}$$

WANT to ADD SINUSOIDS

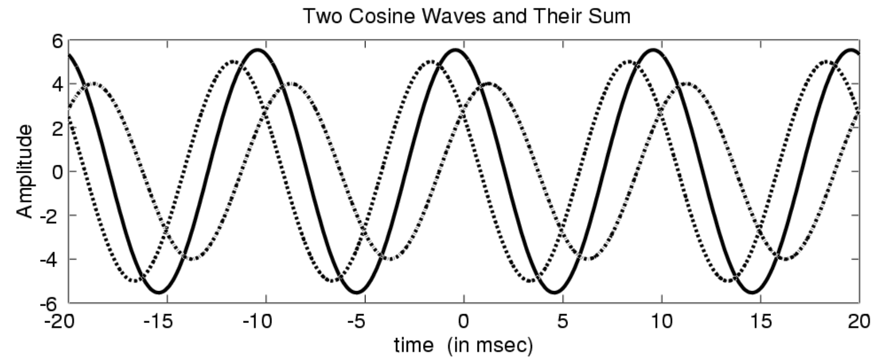
- ALL SINUSOIDS have SAME FREQUENCY
- HOW to GET {Amp,Phase} of RESULT ?

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$
$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

ADD SINUSOIDS

- Sum Sinusoid has **SAME** Frequency



PHASOR ADDITION RULE

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$
$$= A\cos(\omega_0 t + \phi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^{N} A_k e^{j\phi_k} = A e^{j\phi}$$

Phasor Addition Proof

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \sum_{k=1}^{N} \Re e\left\{A_k e^{j(\omega_0 t + \phi_k)}\right\}$$
$$= \Re e\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\}$$
$$= \Re e\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{j\omega_0 t}\right\}$$
$$= \Re e\left\{\left(A e^{j\phi}\right) e^{j\omega_0 t}\right\} = A\cos(\omega_0 t + \phi)$$

POP QUIZ: Add Sinusoids

- ADD THESE 2 SINUSOIDS:

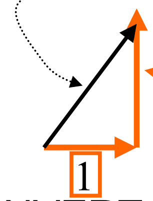
$$x_1(t) = \cos(77\pi t)$$
$$x_2(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- COMPLEX ADDITION:

$$\boxed{1e^{j0} + \sqrt{3}e^{j0.5\pi}}$$

POP QUIZ (answer)

- COMPLEX ADDITION: $\boxed{1 + j\sqrt{3} = 2e^{j\pi/3}}$



$$\boxed{j\sqrt{3} = \sqrt{3}e^{j0.5\pi}}$$

- CONVERT back to cosine form:

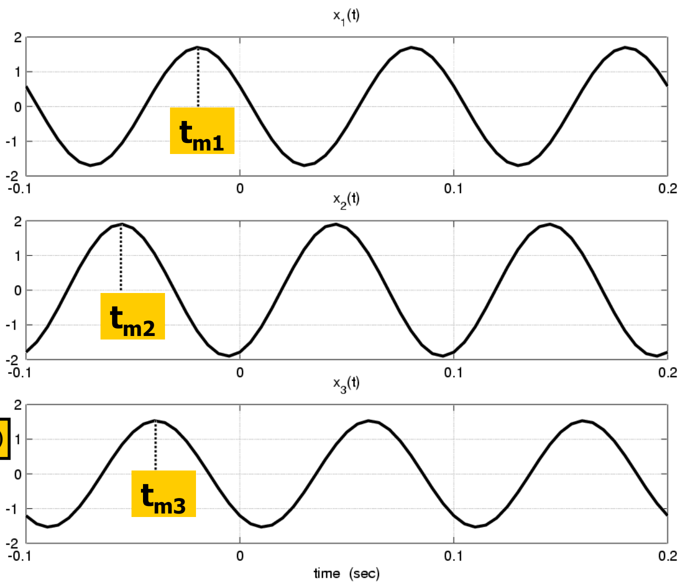
$$\boxed{x_3(t) = 2\cos(77\pi t + \tfrac{\pi}{3})}$$

ADD SINUSOIDS EXAMPLE

$x_1(t)$

$x_2(t)$

$x_3(t) = x_1(t) + x_2(t)$

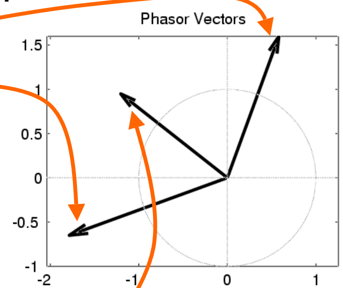


Convert Time-Shift to Phase

- Measure peak times:
  - t_{m1}=-0.0194, t_{m2}=-0.0556, t_{m3}=-0.0394
- Convert to phase (T=0.1)
  - ϕ_1=$-\omega t_{m1}$ = $-2\pi(t_{m1}/T)$ = $70\pi/180$,
  - ϕ_2= $200\pi/180$
- Amplitudes
  - A_1=1.7, A_2=1.9, A_3=1.532

Phasor Add: Numerical

- Convert Polar to Cartesian
  - $X_1 = 0.5814 + j1.597$
  - $X_2 = -1.785 - j0.6498$
  - sum =
  - $X_3 = -1.204 + j0.9476$
- Convert back to Polar
  - $X_3 = 1.532$ at angle $141.79\pi/180$
  - This is the sum



ADD SINUSOIDS

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VECTOR (PHASOR) ADD