SOLUTIONS P.S. #4

Problem 4.1 a) Decoding the MATLAB code:

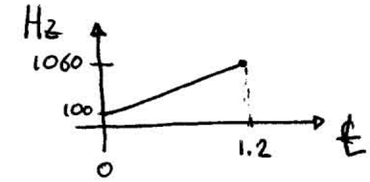
$$\psi(t) = 800\pi t^2 + 200\pi t + \frac{\pi}{2}, \quad \text{with } t \text{ from } 0 \text{ to } 1.2$$

$$x(t) = \mathrm{Re}\left(e^{j\psi(t)}\right)$$

$$\omega(t) = \frac{d\psi(t)}{dt} = 1600\pi t + 200\pi$$

Hence: $t=0 \rightarrow \omega = 200\pi$ or $f = 100\,Hz$

$t = 1.2 \rightarrow \omega = 1920\pi + 200\pi = 2120\pi$ or $f = 1060\,Hz$



b) Analysis of the problem:

$$2at + b = 3000 \cdot 2\pi \quad \text{at } t=0 \longrightarrow b = 6000\pi$$

$$= 600 \cdot 2\pi \quad \text{at } t = 0.8 \rightarrow 2a \times 0.8 + b = 1200\pi$$

$$\therefore\ a = \frac{1200\pi - 6000\pi}{1.6} = -\frac{4800\pi}{1.6} = -3000\pi$$

Code:

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tt = 0 : 0.0001 : 0.8;
aa = -3000*pi;
bb= 6000*pi ;
psi = aa*tt.*tt + bb*tt ;
xx = real (exp(j*psi));
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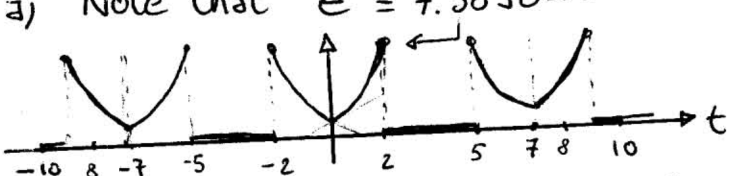
Problem 4.2: a) Note that $e^2 = 7.3890\cdots$



b) $$\frac{1}{7}\int_{-2}^{5} x(t)\,dt = \frac{1}{7}\int_{-2}^{0} e^{-t}\,dt + \frac{1}{7}\int_{0}^{2} e^{t}\,dt = \frac{2}{7}\int_{0}^{2} e^{t}\,dt = (e^2 - 1)\frac{2}{7} = 1.825\cdots$$

Problem 4.3

a) $$a_k = \frac{1}{7}\int_{-2}^{5} x(t)\, e^{-j\frac{2\pi}{7}kt}\,dt = \frac{1}{7}\int_{-2}^{2} e^{|t|}\, e^{-j\frac{2\pi}{7}kt}\,dt$$

Note: The integral does not necessarily have to be taken from 0 to T (the period), but it shoud span an entire period. (What you may be missing at the end is added back to the front for a periodic signal!)

b) $$a_k = \frac{1}{7}\left[\int_{0}^{2} e^{t} e^{-j\frac{2\pi kt}{7}}\,dt + \int_{-2}^{0} e^{-t} e^{-j\frac{2\pi kt}{7}}\,dt\right]$$

$$= \frac{1}{7}\int_{0}^{2} e^{t}\left(e^{-j\frac{2\pi kt}{2}} + e^{+j\frac{2\pi kt}{2}}\right)dt \qquad (*)$$

$$= \frac{2}{7}\int_{0}^{2} e^{t}\cos\frac{2\pi kt}{7}\,dt$$

If you know that $\int_{0}^{T} e^{at}\cos bt\,dt = \dfrac{a e^{aT}\cos bT + b e^{aT}\sin bT - a}{a^2+b^2}$

proceed to (letting $T=2$, $a=1$ and $b=\frac{2\pi k}{7}$)

$$a_k = \frac{14}{49+4\pi^2k^2}\left[e^2\cos\left(\frac{4\pi k}{7}\right) + \frac{2\pi k}{7}\, e^2\sin\left(\frac{4\pi k}{7}\right) - 1\right]$$

Alternatively (if you're stuck on an island without integral tables) proceed as follows:

$$(*) = \frac{1}{7}\left(\int_{0}^{2} e^{\left(1-j\frac{2\pi k}{7}\right)t}\,dt + \int_{0}^{2} e^{\left(1+j\frac{2\pi k}{7}\right)t}\,dt\right)$$

$$= 2\,\mathrm{Re}\,\frac{1}{7}\int_{0}^{2} e^{\left(1-j\frac{2\pi k}{7}\right)t}\,dt = \frac{2}{7}\,\mathrm{Re}\,\frac{e^{\left(1-j\frac{2\pi k}{7}\right)2} - 1}{1-j\frac{2\pi k}{7}}$$

$$= \frac{2}{7}\,\mathrm{Re}\,\frac{e^2\cos\left(\frac{4\pi k}{7}\right) - 1 - j e^2\sin\left(\frac{4\pi k}{7}\right)}{1-j\frac{2\pi k}{7}}$$

$$= \frac{2}{7}\,\frac{\left(e^2\cos\frac{4\pi k}{7} - 1\right) + \frac{2\pi k}{7} e^2\sin\frac{4\pi k}{7}}{1+4\pi^2k^2/49} = \frac{14}{49+4\pi^2k^2}\left[e^2\cos\left(\frac{4\pi k}{7}\right) + \frac{2\pi k}{7} e^2\sin\left(\frac{4\pi k}{7}\right) - 1\right]$$

Problem 4.4: $x(t) = \cos^3(22\pi t - \frac{\pi}{2})$

Note first that $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$, thus

$$\cos^3\theta = \left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)^3 = \frac{e^{3j\theta}}{8} + \frac{3}{8} e^{j\theta} + \frac{3}{8} e^{-j\theta} + \frac{1}{8} e^{-3j\theta} \qquad (*)$$

(a) With $\theta = 22\pi t$, the fundamental frequency is

$$\omega_0 = 22\pi \longrightarrow f_0 = 11 \text{ Hz}$$

(b) From $(*)$, with noting that phase $3(\mp\frac{\pi}{2})$ is equivalent to phase $\pm\frac{\pi}{2}$

$$x(t) = \frac{1}{8} e^{j(66\pi t + \frac{\pi}{2})} + \frac{3}{8} e^{j(22\pi t - \frac{\pi}{2})} + \frac{3}{8} e^{j(-22\pi t + \frac{\pi}{2})} + \frac{1}{8} e^{j(-66\pi t - \frac{\pi}{2})}$$

Hence: (without integration:)

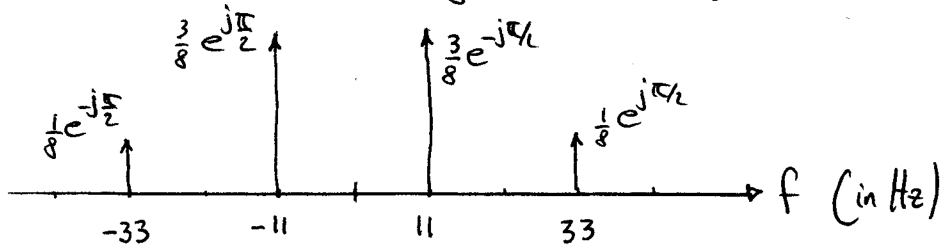
$$a_0 = 0, \quad a_1 = \frac{3}{8} e^{-j\pi/2}, \quad a_2 = 0, \quad a_3 = \frac{1}{8} e^{j\pi/2}$$

$$a_{-1} = \frac{3}{8} e^{j\pi/2}, \quad a_{-2} = 0, \quad a_{-3} = \frac{1}{8} e^{-j\pi/2}$$

all other are zero.

In 'cartesian form': $a_1 = -j\frac{3}{8}$, $a_3 = j\frac{1}{8}$ all other zero.

$a_{-1} = j\frac{3}{8}$, $a_3 = -j\frac{1}{8}$



problem 4.5

(a) Explain why the ratio of the frequencies of successive notes must be $2^{1/12}$

Let the ratio be $r = 2^n$, n is a real number

$$r^{12} = 2 = 2^1$$

$$= (2^n)^{12} = 2^{12n}$$

$$12n = 1 \qquad n = \frac{1}{12}$$

$$r = 2^{1/12}$$

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(b)

C	$C^\#$	D	E^b	E	F	$F^\#$	G	$G^\#$	A	B^b	B	C
40	41	42	43	44	45	46	47	48	49	50	51	52
262	277	294	311	330	349	370	392	415	440	466	494	523

$$C = 440 * 2^{40-49} = 261.6256 = 262\ Hz$$

$$C^\# = 440 * 2^{41-49} = 277\ Hz$$

⋮

(c) The simple formula is:

$$f_n = 440 \times 2^{(n-49)/12}$$

$$f_n = 440 \times 2^{(n-49)/12}$$

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