Problem 5.1

$$x(t) = \cos^4(100\pi t + 0.25\pi)$$

(a) Determine the fundamental frequency of the signal

Let us write $x(t) = \left[\frac{1}{2}e^{j(100\pi t+0.25\pi)} + \frac{1}{2}e^{-j(100\pi t+0.25\pi)}\right]^4$

and recall that

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac$$

Therefore,

$$x(t) = \left[\frac{1}{4}e^{j(200\pi t+0.5\pi)} + \frac{1}{4}e^{-j(200\pi t+0.5\pi)} + \frac{1}{2}\right]^2$$

$$= \frac{1}{16}e^{j(400\pi t+\pi)} + \frac{1}{16}e^{-j(400\pi t+\pi)} + \frac{1}{4}$$

$$+\frac{1}{8} + \frac{1}{4}e^{j(200\pi t+0.5\pi)} + \frac{1}{4}e^{-j(200\pi t+0.5\pi)}$$

$$= \frac{3}{8} + \frac{1}{4}e^{j(200\pi t+0.5\pi)} + \frac{1}{4}e^{-j(200\pi t+0.5\pi)}$$

$$+\frac{1}{16}e^{j(400\pi t+\pi)} + \frac{1}{16}e^{-j(400\pi t+\pi)}$$

Fundamental frequency of $x(t)$ is $\omega_0 = 200\pi$ rads/sec

or $f_0 = \frac{\omega_0}{2\pi} = 100$ Hz

(b) Determine the Fourier series coefficients of the signal

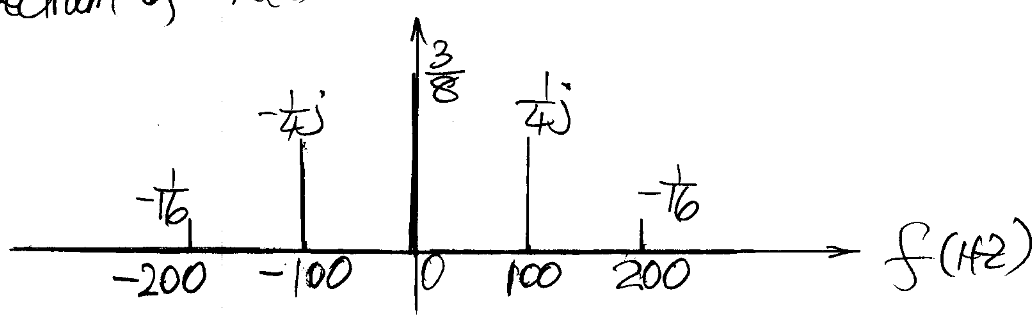
$$x(t) = \sum_k a_k e^{jk\cdot 200\pi\cdot t}$$

Where $a_0 = \frac{3}{8}$, $a_1 = \frac{1}{4}e^{j0.5\pi} = \frac{1}{4}j$, $a_{-1} = \frac{1}{4}e^{-j0.5\pi} = -\frac{1}{4}j$,

$a_2 = \frac{1}{16}e^{j\pi} = -\frac{1}{16}$, $a_2 = \frac{1}{16}e^{-j\pi} = -\frac{1}{16}$.

All other a_k's are zero.

(c) Spectrum of $x(t)$



Problem 5.2

$T_0 = 4$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{4}kt}$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\frac{2\pi}{T_0}kt} dt = \frac{1}{4}\int_0^1 e^{-3t} e^{-j\frac{2\pi}{4}kt} dt$$

Hence,

$$x(t) = \begin{cases} e^{-3t}, & 0 \le t < 1 \\ 0, & 1 \le t < 4, \end{cases} \qquad x(t+4) = x(t), \ \forall t$$

(a) $a_k = \frac{1}{4}\int_0^1 e^{-3t}\cdot e^{-j\frac{2\pi}{4}kt}\,dt = \frac{1}{4}\int_0^1 e^{(-3-j\frac{\pi}{2}k)t}\,dt$

$$= \frac{1}{4}\cdot\frac{e^{(-3-j\frac{\pi}{2}k)t}\Big|_{t=0}^{1}}{-3-j\frac{\pi}{2}k} = \frac{1}{4}\cdot\frac{e^{-3-j\frac{\pi}{2}k}-1}{-3-j\frac{\pi}{2}k} = \frac{1}{4}\cdot\frac{1-e^{-3}\cdot e^{-j\frac{\pi}{2}k}}{3+\frac{\pi}{2}k\cdot e^{j\frac{\pi}{2}}}$$

$$|a_k| = \sqrt{a_k a_k^*} = \frac{0.25\sqrt{1+e^{-9}-2e^{-3}\cos(\frac{\pi}{2}k)}}{\sqrt{9+\frac{\pi^2}{4}k^2}}$$

(b) Substitute with $k=0, 1, 2, 3$, we find

$a_0 = 0.0792$

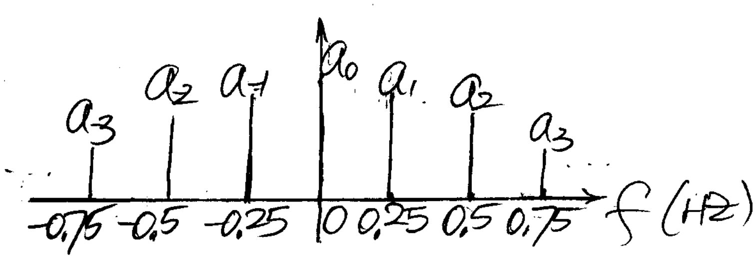
$a_1 = 0.0739\cdot e^{-j0.4326} \Rightarrow a_{-1} = 0.0739\cdot e^{j0.4326}$

$a_2 = 0.0604\, e^{-j0.8084} \Rightarrow a_{-2} = 0.0604\, e^{j0.8084}$

$a_3 = 0.0448\, e^{-j1.0536} \Rightarrow a_3 = 0.0448\, e^{j1.0536}$

...

Spectrum $\quad f_0 = \frac{1}{T_0} = 0.25\ \text{Hz}$



Problem 5.3

(a) The fundamental frequency

$$f_0 = \gcd(60, 150) = 30\ Hz$$

(b)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

$$\omega_0 = 2\pi f_0 = 60\pi \text{ rads/sec}$$

Non-zero Fourier series coefficients.

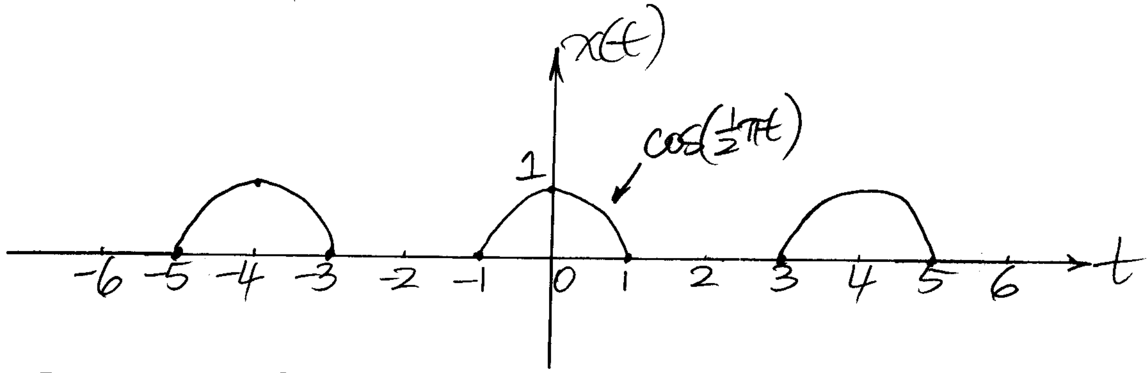
k	a_k
0	$12e^{j\pi} = -12$
2	$7e^{-j\pi/4}$
-2	$7e^{j\pi/4}$
5	$5e^{-j\pi/3}$
-5	$5e^{j\pi/3}$

Problem 5.4

$$x(t)=\begin{cases}\cos(\frac{1}{2}\pi t), & -1\le t\le 1\\ 0, & -2\le t<-1,\ 1<t<2\end{cases}$$

$T_0=4$

(a) Sketch the periodic function $x(t)$ for $-6\le t<6$



(b) Determine a_0

$$a_0=\frac{1}{T_0}\int_0^{T_0}x(t)\,dt=\frac{1}{4}\int_0^4 x(t)\,dt$$

$$=\frac{1}{4}\int_{-1}^{1}\cos(\tfrac{1}{2}\pi t)\,dt=\frac{1}{4}\cdot\frac{1}{\frac{1}{2}\pi}\sin(\tfrac{1}{2}\pi t)\Big|_{t=-1}^{1}$$

$$=\frac{1}{2\pi}\cdot(1-(-1))=\frac{1}{\pi}$$

(c) $$a_k=\frac{1}{4}\int_{-1}^{1}\cos(\tfrac{1}{2}\pi t)\,e^{-jk\frac{2\pi}{4}t}\,dt$$

(d) $$a_k=\frac{1}{4}\int_{-1}^{1}\left(\frac{1}{2}e^{j\frac{\pi}{2}t}+\frac{1}{2}e^{-j\frac{\pi}{2}t}\right)e^{-j\frac{\pi}{2}kt}\,dt$$

$$=\frac{1}{4}\int_{-1}^{1}\left(\frac{1}{2}e^{j\frac{\pi}{2}(1-k)t}+\frac{1}{2}e^{-j\frac{\pi}{2}(1+k)t}\right)dt$$

$$a_k = \frac{1}{4}\cdot\left[\frac{1}{2}\cdot\frac{1}{j\frac{\pi}{2}(1-k)}\,e^{j\frac{\pi}{2}(1-k)t}\Big|_{t=-1}^{1}\right.$$

$$\left.+\frac{1}{2}\cdot\frac{1}{-j\frac{\pi}{2}(1+k)}\,e^{-j\frac{\pi}{2}(1+k)t}\Big|_{t=-1}^{1}\right]$$

$$=\frac{1}{4}\;\frac{2j\sin\left(\frac{\pi}{2}(1-k)\right)}{j\pi(1-k)}+\frac{1}{4}\cdot\frac{-2j\sin\left(\frac{\pi}{2}(1+k)\right)}{-j\pi(1+k)}$$

$$=\frac{1}{2}\frac{\sin\left(\frac{\pi}{2}(1-k)\right)}{\pi(1-k)}+\frac{1}{2}\cdot\frac{\sin\left(\frac{\pi}{2}(1+k)\right)}{\pi(1+k)}$$

Special cases: $k=0$: $a_k=\frac{1}{2}\cdot\frac{\sin(-\frac{\pi}{2})}{-\pi}+\frac{1}{2}\cdot\frac{\sin\frac{\pi}{2}}{\pi}=\frac{1}{\pi}$

When $k=1$: $\frac{\sin\left(\frac{\pi}{2}(1+k)\right)}{\pi(1+k)}=0$. Recall that $\lim_{c\to 0}\frac{\sin c}{c}=1$.

Hence, $\frac{1}{2}\frac{\sin\left(\frac{\pi}{2}(1-k)\right)}{\pi(1-k)}\Big|_{k=1}=\frac{\sin\left(\frac{\pi}{2}(1-k)\right)}{\frac{\pi}{2}(1-k)}\Big|_{k=1}\cdot\frac{1}{4}=\frac{1}{4}$

Therefore, $a_1=\frac{1}{4}+0=\frac{1}{4}$

$a_{-1}=a_1^*=\frac{1}{4}$

When $k=$ odd but $k\neq\pm1$, $\sin\left(\frac{\pi}{2}(1\pm k)\right)=0$

$k+1=$ even

thus $a_k=0$

When k=even, we need to consider two cases.

If $k=4m$, $m\in\mathbb{Z}$ (k is divisible by 4)

$$\sin\left(\frac{\pi}{2}(1-k)\right)=\sin\left(\frac{\pi}{2}-2m\pi\right)=\sin\left(\frac{\pi}{2}\right)=1$$

$$\sin\left(\frac{\pi}{2}(1+k)\right)=\sin\left(\frac{\pi}{2}+2\pi m\right)=\sin\left(\frac{\pi}{2}\right)=1$$

Hence, $a_k=\frac{1}{2}\cdot\frac{1}{\pi(1-k)}+\frac{1}{2}\cdot\frac{1}{\pi(1+k)}=\frac{1}{\pi}\cdot\frac{1}{1-k^2}$

If $k=4m+2$, $m\in\mathbb{Z}$ (k is even but not divisible by 4)

$$\sin\left(\frac{\pi}{2}(1-k)\right)=\sin\left(\frac{\pi}{2}-2\pi m-\pi\right)=\sin\left(-\frac{\pi}{2}\right)=-1$$

$$\sin\left(\frac{\pi}{2}(1+k)\right)=\sin\left(\frac{\pi}{2}+2\pi m+\pi\right)=\sin\left(\frac{3\pi}{2}\right)=-1$$

Hence, $a_k=\frac{1}{2}\cdot\frac{-1}{\pi(1-k)}+\frac{1}{2}\cdot\frac{-1}{\pi(1+k)}=\frac{1}{\pi}\cdot\frac{1}{k^2-1}$

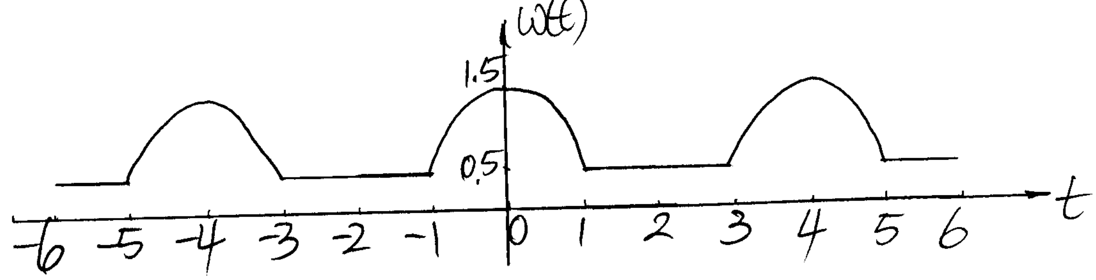
In Summary,

$$a_k=\begin{cases}\frac{1}{4}, & k=\pm1\\ \frac{1}{\pi}\cdot\frac{1}{1-k^2}, & k \text{ is divisible by 4 (includes } k=0)\\ \frac{1}{\pi}\cdot\frac{1}{k^2-1}, & k \text{ is even but not divisible by 4}\\ 0, & k \text{ is odd but } k\neq\pm1\end{cases}$$

Problem 5.5

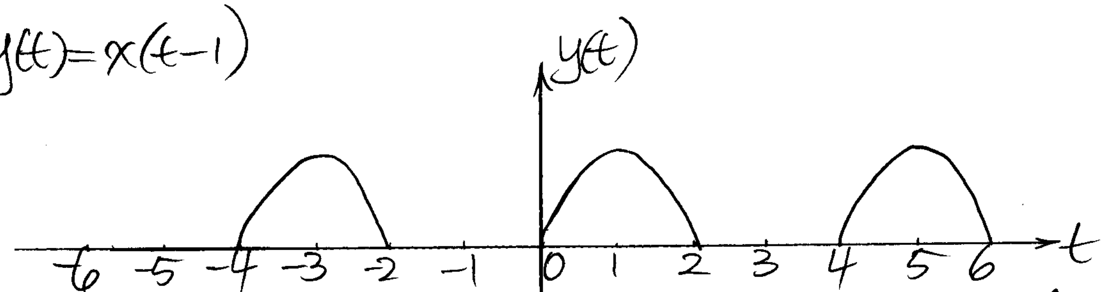
$x(t)$ is periodic with period $T_0=4$. $x(t)=\begin{cases}\cos(\frac{\pi}{2}t), & -1<t\le 1\\ 0, & -2\le t<-1,\ 1<t<2\end{cases}$

(a) $w(t)=x(t)+\frac{1}{2}$. Plot $w(t)$



(b) Since $w(t)$ and $x(t)$ are different only by a constant $\frac{1}{2}$, we must have $b_k=\begin{cases}a_k, & \text{if } k\ne 0\\ a_0+\frac{1}{2}, & \text{if } k=0\end{cases}$

(c) $y(t)=x(t-1)$



(d) From $x(t)=\sum_{k=-\infty}^{\infty}a_k e^{j\frac{\pi}{2}k\cdot t}$, we find $x(t-1)=\sum_{k=-\infty}^{\infty}a_k e^{j\frac{\pi}{2}(t-1)\cdot k}$

$=\sum_{k=-\infty}^{\infty}\left(a_k e^{-j\frac{\pi}{2}k}\right)\cdot e^{j\frac{\pi}{2}t}$

Therefore, $c_k=a_k e^{-jk\frac{\pi}{2}}$. In particular, $c_0=a_0$.

(e) From (d), we find $|c_k|=|a_k|$.