ECE 2025 - Solutions - Problem Set #2

Note Title 1/21/2005

2.1) Given $z(t) = 33e^{j7.5\pi t}$

a) Evaluate: $\int_{-1}^{1} z(t)\,dt$

Remember: $\int e^{at}dt = \frac{1}{a}e^{at}$

$\Rightarrow \int_{-1}^{1} 33e^{j7.5\pi t}dt \Rightarrow 33\int_{-1}^{1} e^{j7.5\pi t}dt$ *let $a = j7.5\pi$

$= 33\int_{-1}^{1} e^{at}dt$

$= 33\cdot\left[\frac{1}{a}e^{at}\right]_{-1}^{1}$

$= 33\cdot\left[\frac{1}{j7.5\pi}\cdot e^{j7.5\pi t}\right]_{-1}^{1}$

$= \frac{33}{j7.5\pi}\cdot\left[e^{j7.5\pi} - e^{-j7.5\pi}\right]$

*$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$\therefore 2j\sin\theta = e^{j\theta} - e^{-j\theta}$

$= \frac{33}{\cancel{j}7.5\pi}\cdot\left[2\cancel{j}\cdot\sin(7.5\pi)\right]$

$= \frac{66}{7.5\pi}\cdot\sin(\cancel{7.5}\pi)^{-1}$

$= -\frac{66}{7.5\pi}$

$\boxed{= -2.8011}$ ⇐ Solution

2.1)

b) Determine all values of u for:

$$\int_{-u}^{u} z(t)\,dt = 0 \quad , u > 0$$

$$\int_{-u}^{u} 33e^{j7.5\pi t}\,dt = 33\int_{-u}^{u} e^{j7.5\pi t}\,dt$$

From part a., we have

$$= \frac{33}{j7.5\pi}\left[e^{j7.5\pi t}\right]_{-u}^{u}$$

$$= \frac{33}{j7.5\pi}\left[e^{j7.5\pi u} - e^{-j7.5\pi u}\right]$$

$*\ \sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$2j\sin\theta = e^{j\theta} - e^{-j\theta}$

$$= \frac{33}{j7.5\pi}\left[2j\sin(7.5\pi u)\right] = 0$$

$$\therefore \Rightarrow \sin(7.5\pi u) = 0$$

The question is reduced to finding all instants where $\sin(\theta) = 0$. For $\sin(\theta) = 0$, θ must be an integer multiple of π i.e., $\theta = \pi \cdot K$, for all integer K.

Above: $\theta = 7.5\pi u = \frac{15}{2}\pi u = \pi$

$$\Rightarrow u = \frac{2}{15}\cdot K$$

Solution $\Rightarrow$ $\boxed{\therefore \text{ for } u > 0,\ u = \frac{2}{15}\cdot K \text{ for } K \geq 1}$

i.e. positive integer multiples of $\frac{2}{15}$

2.1)
c) Evaluate $\int_{-1}^{1} z^*(t)\, z(t)\, dt$

Remember: $z^*(t)\, z(t) = |z(t)|^2$

We are given $z(t) = 33\, e^{j7.5\pi t}$

$\therefore\ |z(t)| = |33 e^{j7.5\pi t}| = 33$

and $|z(t)|^2 = (33)^2$

$$\Rightarrow \int_{-1}^{1} z^*(t) z(t)\, dt = \int_{-1}^{1} |z(t)|^2 dt = \int_{-1}^{1} (33)^2 dt$$

$$= (33)^2 [t]_{-1}^{1}$$

$$= (33)^2 (1 - -1)$$

$$= (33)^2 \cdot 2$$

$\boxed{= 2178}$ ⇐ Solution

22) Express as $x(t) = A\cos(\omega_0 t + \phi)$

let $X = Ae^{j\phi}$

a) $x_a(t) = 3\cos(33\pi t - 5\pi/6) + 3\cos(33\pi t - \pi/6)$

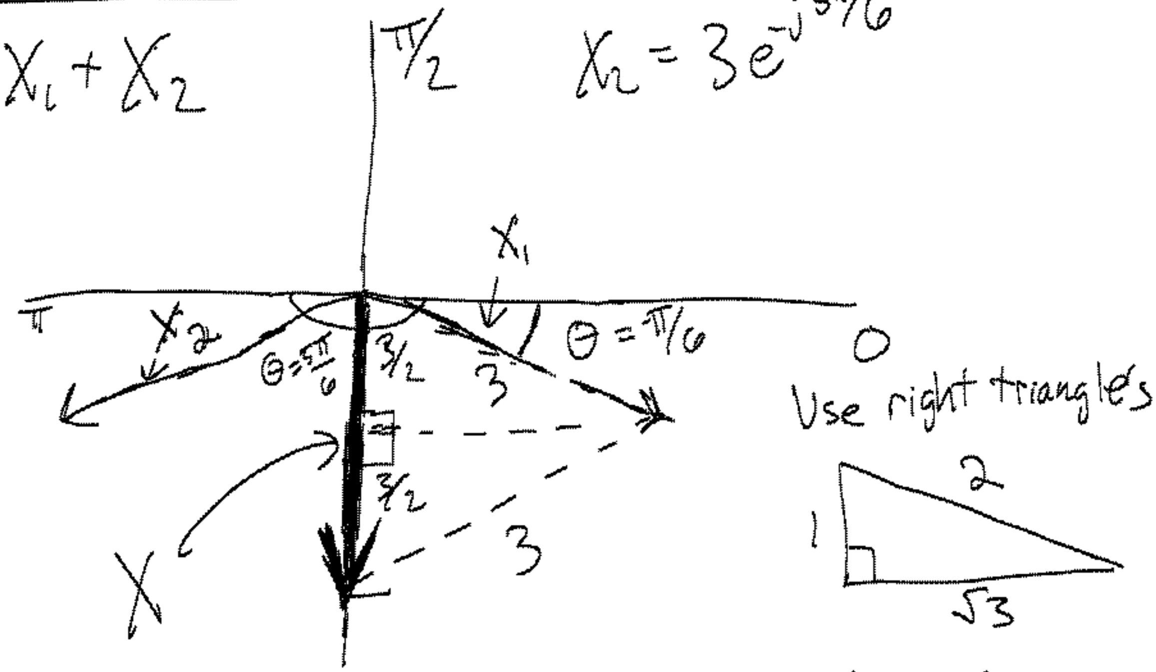
$\therefore X = 3e^{-j5\pi/6} + 3e^{-j\pi/6}$.

<u>Vector diagram</u>

$X_1 = 3e^{-j\pi/6}$

$X_2 = 3e^{-j5\pi/6}$

$X = X_1 + X_2$



- Add vector X_1 to the end of vector X_2
- $X = 3e^{-j\pi/2} \Rightarrow A = 3,\ \phi = -\pi/2$

$\boxed{\therefore x_a(t) = 3\cos(33\pi t - \pi/2)}$ ← solution

2.2)

a) <u>continued</u>

<u>Phasor addition</u>

let $X = 3e^{-j5\pi/6} + 3e^{-j\pi/6} = Ae^{j\phi}$.

and $x_a(t) = A\cos(w_0 t + \phi)$, $w_0 = 33\pi$

must convert to rectangular:

$3e^{-j5\pi/6} \Rightarrow 3\cos(-5\pi/6) + j3\sin(-5\pi/6)$

$= \underline{-\frac{3\sqrt{3}}{2} - j\frac{3}{2}}$

$3e^{-j\pi/6} \Rightarrow 3\cos(-\pi/6) + j3\sin(-\pi/6)$

$= \underline{\frac{3\sqrt{3}}{2} - j\frac{3}{2}}$

$X = \left(\frac{-3\sqrt{3}}{2} - j\frac{3}{2}\right) + \left(\frac{3\sqrt{3}}{2} - j\frac{3}{2}\right)$

$= -j3 = 3e^{-j\pi/2}$

$$\boxed{\therefore x(t) = 3\cos(33\pi t - \pi/2)}$$ esolution

2.2)

b) $x_b(t) = 33\cos(333\pi t + 33\pi) +$

$33\sqrt{2}\cos(333\pi t - 33.25\pi) +$

$33\sqrt{2}\cos(333\pi t + 33.25\pi)$

$$X = 33e^{j33\pi} + 33\sqrt{2}e^{-j33.25\pi} + 33\sqrt{2}e^{j33.25\pi} = Ae^{j\phi}\ ;\ w_0 = 333\pi$$

- phases can be reduced to be in the range $-\pi < \phi \leq \pi$ by adding/subtracting 2π (i.e., $\phi \pm 2\pi k$)

33π : $33\pi - k \cdot 2\pi$

$33\pi - (16)\cdot 2\pi = \underline{\pi}$

-33.25π : $-33.25\pi + k\cdot 2\pi$

$-33.25\pi + (17)\cdot 2\pi = \underline{0.75\pi}$

33.25π : $33.25\pi - k\cdot 2\pi$

$33.25\pi - (17)\cdot 2\pi = \underline{-0.75\pi}$

$$\therefore X = 33e^{j\pi} + 33\sqrt{2}e^{j0.75\pi} + 33\sqrt{2}e^{-j0.75\pi}$$

2.2)

b) (continued)

$$X = 33e^{j\pi} + 33\sqrt{2}\,e^{j0.75\pi} + 33\sqrt{2}\,e^{-j0.75\pi}$$

Vector diagram

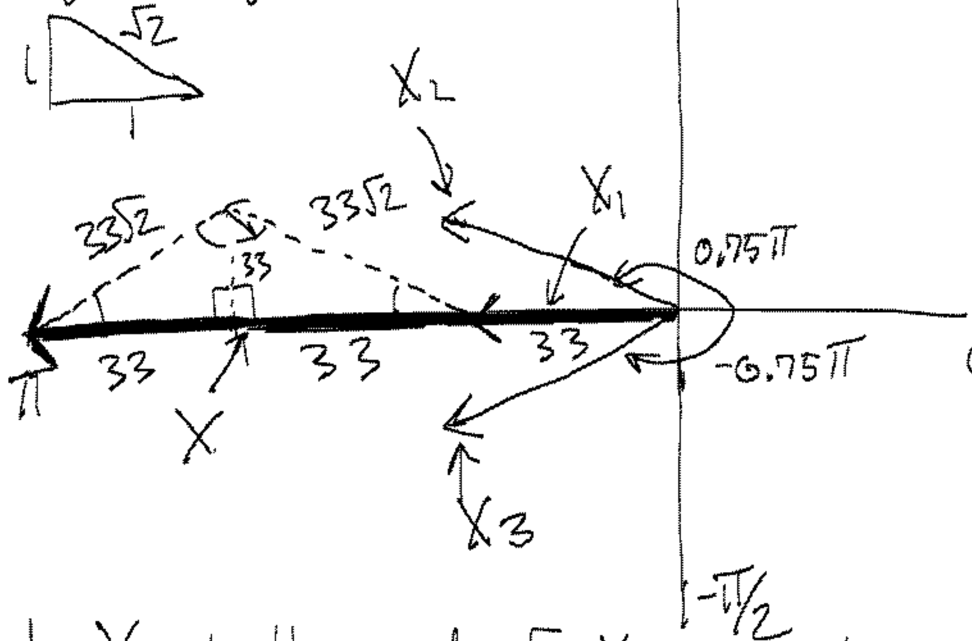
$X_1 = 33e^{j\pi}$

$X_2 = 33\sqrt{2}\,e^{j0.75\pi}$

$X_3 = 33\sqrt{2}\,e^{-j0.75\pi}$

$$X = X_1 + X_2 + X_3$$

use right triangles



- add X_2 to the end of X_1 and then add X_3
- $\phi = \pi$, $A = 99$
- $X = 99e^{j\pi}$

$$\boxed{\therefore\ x_b(t) = 99\cos(333\pi t + \pi)}$$

solution

2.2
b) (continued)

Phasor Addition

let $X = 33e^{j\pi} + 33\sqrt{2}e^{j0.75\pi} + 33\sqrt{2}e^{-j0.75\pi} = Ae^{j\phi}$

and $x_b(t) = A\cos(\omega_0 t + \phi)$

Convert to rectangular

$33e^{j\pi}: 33\cos(\pi) + j33\cancel{\sin(\pi)}^{0} = -33$

$33\sqrt{2}e^{j0.75\pi}: 33\sqrt{2}\cos(0.75\pi) + j33\sqrt{2}\sin(0.75\pi) = -33 + j33$

$33\sqrt{2}e^{-j0.75\pi}: 33\sqrt{2}\cos(-0.75\pi) + j33\sqrt{2}\sin(-0.75\pi) = -33 - j33$

$= -99$

$X = -99 = 99e^{j\pi}$

$\boxed{\therefore x_b(t) = 99\cos(333\pi t + \pi)}$ ← solution

2.2)

c) $x_c(t) = \sum_{k=0}^{4} 2k \cos(3333\pi t - (2k+1)\pi/4)$

$$X = \sum_{k=0}^{4} 2k e^{-j(2k+1)\pi/4} = Ae^{j\phi}$$

$$X = 0 + 2e^{-j3\pi/4} + 4e^{-j5\pi/4} + 6e^{-j7\pi/4} + 8e^{-j9\pi/4}.$$

$\phi + 2\pi$ $\phi + 2\pi$ $\phi + 2\pi$

let $-\pi < \phi \le \pi$

$$X = 2e^{-j3\pi/4} + 4e^{j3\pi/4} + 6e^{j\pi/4} + 8e^{-j\frac{\pi}{4}}.$$

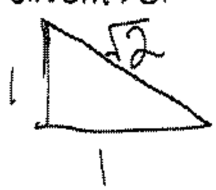
Vector Diagram

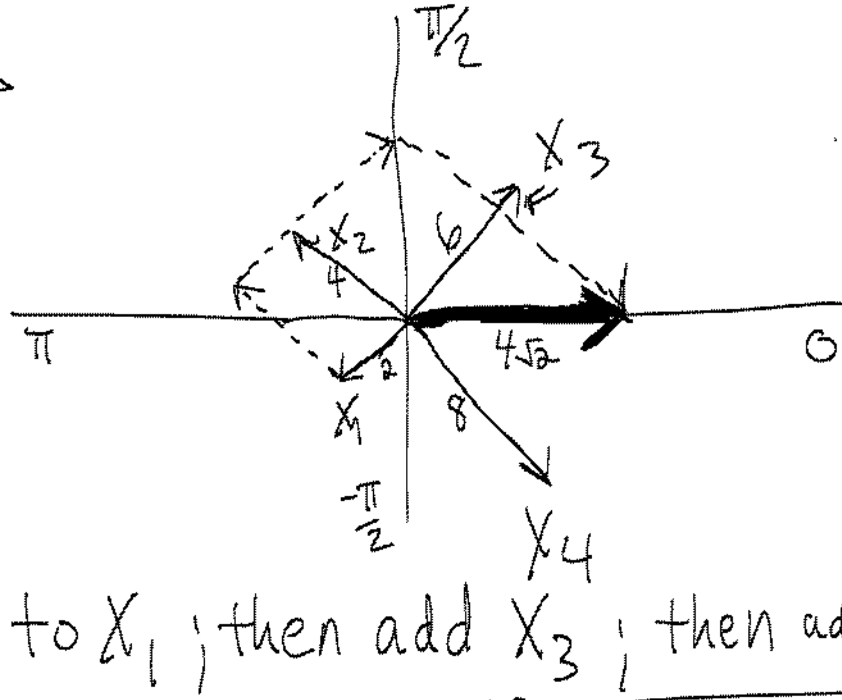
$X_1 = 2e^{-j3\pi/4}$ $X_4 = 8e^{-j\pi/4}$

$X_2 = 4e^{j3\pi/4}$

$X_3 = 6e^{j\pi/4}$

Remember





- Add X_2 to X_1; then add X_3; then add X_4
- $A = 4\sqrt{2}$, $\phi = 0$; $X = 4\sqrt{2}e^{j0}$ $\boxed{\therefore x_c(t) = 4\sqrt{2}\cos(3333\pi t)}$

2.2)

c.) (continued)

Phasor addition

$$X = 2e^{-j3\pi/4} + 4e^{j3\pi/4} + 6e^{j\pi/4} + 8e^{-j\frac{\pi}{4}}.$$

Convert to rectangular

$2e^{-j3\pi/4}$: $2\cos\left(-\frac{3\pi}{4}\right) + j2\sin\left(-\frac{3\pi}{4}\right) = -\sqrt{2} - j\sqrt{2}$

$4e^{j3\pi/4}$: $4\cos\left(\frac{3\pi}{4}\right) + j4\sin\left(\frac{3\pi}{4}\right) = -2\sqrt{2} + j2\sqrt{2}$

$6e^{j\pi/4}$: $6\cos(\pi/4) + j6\sin(\pi/4) = 3\sqrt{2} + j3\sqrt{2}$

$8e^{-j\pi/4}$: $8\cos(-\pi/4) + j8\sin(-\pi/4) = 4\sqrt{2} - j4\sqrt{2}$

$$X = 4\sqrt{2} + j0$$

- $A = 4\sqrt{2}$ $\quad \phi = 0$
- $X = 4\sqrt{2}e^{j0}$

$$\boxed{\therefore X_c(t) = 4\sqrt{2}\cos(3333\pi t)}$$

2.3) Solve simultaneous equations:

$$10\cos(w_0t - 2\pi/3) = A_1\cos(w_0t+\phi_1) + A_2\cos(w_0t+\phi_2)$$
$$10\cos(w_0t+\pi) = A_1\cos(w_0t+\phi_1) - A_2\cos(w_0t+\phi_2)$$

Convert to phasors:

① $10e^{j2\pi/3} = A_1e^{j\phi_1} + A_2e^{j\phi_2}$

② $10e^{j\pi} = A_1e^{j\phi_1} - A_2e^{j\phi_2}$

If we add ① + ② we get

$$10e^{j2\pi/3} + 10e^{j\pi} = 2A_1e^{j\phi_1}$$

$X_1 = 10e^{j2\pi/3} \Rightarrow -5 - j5\sqrt{3}$

$X_2 = 10e^{j\pi} \Rightarrow -10 - j0$

$$X_1 + X_2 = -15 - j5\sqrt{3} = 17.32e^{-j0.833\pi}$$

If we subtract ① − ② we get

$$10e^{-j2\pi/3} - 10e^{j\pi} = 2A_2e^{j\phi_2}$$

$$X_1 - X_2 = 5 - j5\sqrt{3} = 10e^{-j0.667\pi}$$

Solve

$$X_1 + X_2 = 17.32\, e^{-j0.833\pi} = 2A_1 e^{j\phi_1}$$

$$X_1 - X_2 = 10\, e^{-j0.333\pi} = 2A_2 e^{j\phi_2}$$

$$\left.\begin{array}{l} A_1 = \dfrac{17.32}{2} = 8.66 \\ A_2 = \dfrac{10}{2} = 5 \end{array}\right\} \text{Both } A_1 \text{ and } A_2 \text{ are unique solutions}$$

$$\left.\begin{array}{l} \phi_1 = -0.8333\pi \pm 2\pi k,\ \forall k \text{ integer} \\ \phi_2 = -0.333\pi \pm 2\pi k,\ \forall k \text{ integer} \end{array}\right\} \phi_1 \text{ and } \phi_2 \text{ are } \underline{\text{NOT}} \text{ unique solutions}$$

Problem 2.4: **ECE-2025** **Spring-2005**

```
J = sqrt(-1);
dt = 1/100;
tt = -1 : dt : 1;
Fo = 2;
xx = 300*real( exp( J*(2*pi*Fo*(tt - 0.75) ) ) );
%
subplot(2,1,1)
plot( tt, xx ),   grid
title( 'SECTION of a SINUSOID' ), xlabel('TIME   (sec)')
```

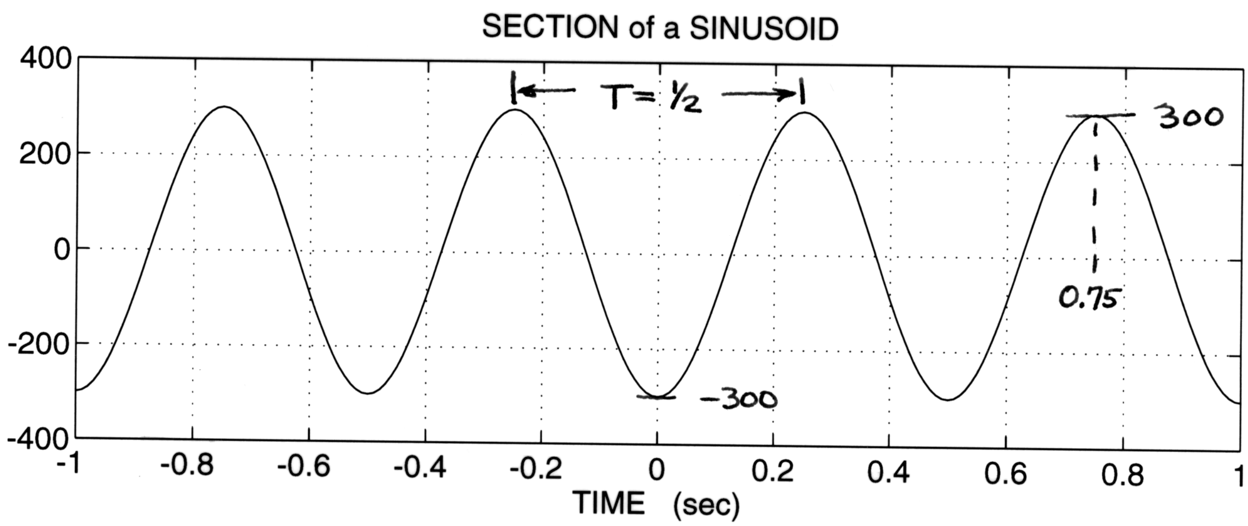
$$x(t) = 300\,\Re e\left\{ e^{j2\pi F_0 (t-0.75)} \right\} \qquad \boxed{F_0 = 2}$$

$$\Rightarrow T = \tfrac{1}{2}\text{ sec}$$

$$= 300\cos\left(4\pi(t - 3/4)\right)$$

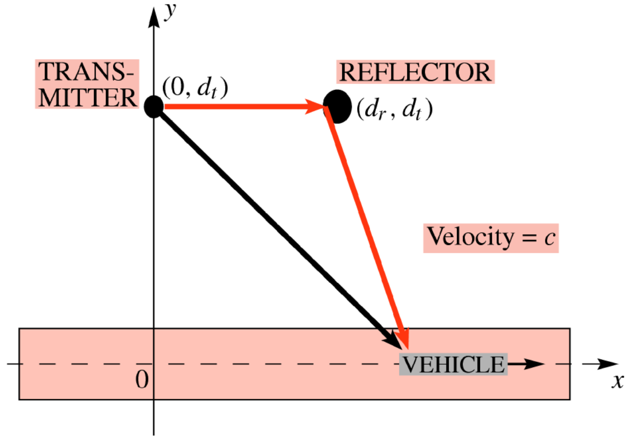
@ $t=0$ $\quad x(0) = 300\cos(-3\pi) = -300$

@ $t = 3/4$ $\quad x(3/4) = 300\cos(0) = 300$ ← POSITIVE PEAK



Problem 2.5: **ECE-2025** **Spring-2005**

In a mobile radio system a transmitting tower sends a sinusoidal signal, and a mobile user receives not one but two copies of the transmitted signal: a direct-path transmission and a reflected-path signal (e.g., from a large building) as depicted in the following figure.



The received signal is the sum of the two copies, and since they travel different distances they have different time delays, i.e.,

$$r(t) = s(t - t_1) + s(t - t_2)$$

The distance between the mobile user in the vehicle at x and the transmitting tower is always changing. Suppose that the direct-path distance is

$$d_1 = \sqrt{x^2 + d_t^2} \quad \text{(meters)}$$

where $d_t = 1000$ meters, and where x is the position of the vehicle moving along the x-axis. Assume that the reflected-path distance is

$$d_2 = d_r + \sqrt{(x - d_r)^2 + d_t^2} \quad \text{(meters)}$$

where $d_r = 55$ meters.

(a) The amount of the delay (in seconds) can be computed for both propagation paths, by converting distance into time delay by dividing by the speed of light ($c = 3 \times 10^8$ m/s).

$$t_1 = d_1/c = \frac{\sqrt{x^2 + d_t^2}}{c} = \frac{\sqrt{x^2 + 10^6}}{3 \times 10^8} \text{ secs.}$$

$$t_2 = d_2/c = \frac{d_r + \sqrt{(x - d_r)^2 + d_t^2}}{c} = \frac{55 + \sqrt{(x - 55)^2 + 10^6}}{3 \times 10^8} \text{ secs.}$$

(b) When the transmitted signal is $s(t) = \cos(300\pi \times 10^6 t)$, the general formula for the received signal is:

$$r(t) = s(t - t_1) + s(t - t_2) = \cos(300\pi \times 10^6 (t - t_1)) + \cos(300\pi \times 10^6 (t - t_2))$$

When $x = 0$ we can calculate t_1 and t_2, and then perform a phasor addition to express $r(t)$ as a sinusoid with a known amplitude, phase, and frequency. When $x = 0$, the time delays are

$$t_1 = \frac{\sqrt{0^2 + 10^6}}{3\times 10^8} = 3.3333\times 10^{-6} \;\text{ secs.}$$

$$t_2 = \frac{55 + \sqrt{(0-55)^2 + 10^6}}{3\times 10^8} = 3.5217\times 10^{-6} \;\text{ secs.}$$

Thus we must perform the following addition:

$$\begin{aligned} r(t) &= \cos(300\pi \times 10^6 (t - 3.3333\times 10^{-6})) + \cos(300\pi \times 10^6 (t - 3.5217\times 10^{-6})) \\ &= \cos(300\pi \times 10^6 t - 1000\pi)) + \cos(300\pi \times 10^6 t - 1056.5113579\pi) \end{aligned}$$

As a phasor addition, we carry out the following steps (since 1000π and 1056π are integer multiples of 2π):

$$\begin{aligned} R &= 1e^{j0} + 1e^{j0.5113579\pi} \\ &= 1 + j0 + (-0.035674 + j0.99936) \\ &= 0.9643 + j0.9994 = 1.389e^{j0.803} = 1.389e^{j0.256\pi} = 1.389\angle 46.02^\circ \end{aligned}$$

From the polar form of the phasor R, we can write $r(t)$ as a sinusoid:

$$r(t) = 1.389\cos(300\pi \times 10^6 t + 0.256\pi)$$

(c) In order to find the locations where the signal strength is zero, we note that the phase of the two delayed sinusoids must differ by an odd multiple of π in order to get cancellation. Thus,

$$\begin{aligned} (2\ell + 1)\pi &= \phi_1 - \phi_2 = -\omega t_1 - (-\omega t_2) \\ &= -300\pi \times 10^6 \left(\frac{\sqrt{x^2 + 10^6}}{3\times 10^8} - \frac{55 + \sqrt{(x-55)^2 + 10^6}}{3\times 10^8} \right) \\ &= -\pi \left(\sqrt{x^2 + 10^6} - 55 - \sqrt{(x-55)^2 + 10^6} \right) \end{aligned}$$

The general solution to this equation is difficult, involving a quartic. However, if we choose $\ell = 27$ so that the left hand side becomes 55π, then the 55π term on the right hand side will cancel, and we obtain an equation in which squaring both sides will produce the answer.

$$\begin{aligned} \pi\sqrt{x^2 + 10^6} &= -\pi\sqrt{(x-55)^2 + 10^6} \\ \Longrightarrow x^2 + 10^6 &= (x-55)^2 + 10^6 \\ \Longrightarrow x^2 &= x^2 - 110x + 55^2 \\ \Longrightarrow 110x &= 55^2 \\ \Longrightarrow x &= \left(\frac{55}{110}\right) 55 = 27.5 \text{ meters} \end{aligned}$$