ECE 2025: Solutions to P.S. #7

Problem 7.1: $y[n] = x[n] + 2x[n-1] - 3x[n-2] - x[n-3] + x[n-5]$

$$\Downarrow$$

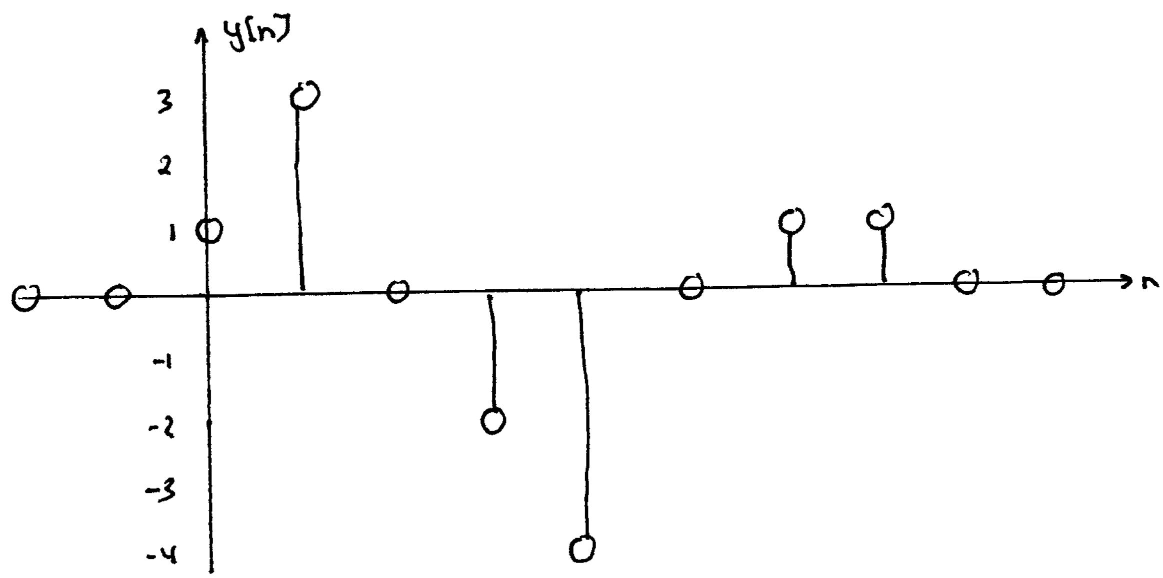
a) $h[n] = \delta[n] + 2\delta[n-1] - 3\delta[n-2] - \delta[n-3] + \delta[n-5]$

b) $b(0) = 1 \quad b(1) = 2 \quad b(2) = -3 \quad b(3) = -1 \quad b(5) = 1$

c) $M = 5$ and $L = 6$

d) With $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ then
$y[n] = h[n] + h[n-1] + h[n-2]$

$n=$	0	1	2	3	4	5	6	7	8
$h[n]$	1	2	-3	-1	0	1	0	0	0
$h[n-1]$	0	1	2	-3	-1	0	1	0	0
$h[n-2]$	0	0	1	2	-3	-1	0	1	0
$y[n]$	1	3	0	-2	-4	0	1	1	0



(a) $x[n] = u[n]$ and $y[n] = u[n-1]$

We need a "delay by one".

$\Rightarrow h[n] = \delta[n-1]$

(b) $x[n] = u[n]$ and $y[n] = \delta[n]$

Since $u[n]$ jumps from 0 to 1 at $n=0$, we need a filter that detects jumps. This can be done with a first-difference filter.

$h[n] = \delta[n] - \delta[n-1]$

(c) $x[n] = (\frac{1}{2})^n u[n]$ and $y[n] = \delta[n-1]$

Use the convolution sum to write linear equations:

$$y[n] = \sum_{k=0}^{M} h[k]\, x[n-k].$$

$y[0] = h[0]\, x[0] + h[1]\, x[-1] + \ldots$ Note: $x[n]=0$ for $n<0$

$0 = h[0] (\frac{1}{2})^0 = h[0] \quad \Rightarrow h[0] = 0$

$y[1] = h[0]\, x[1] + h[1]\, x[0] + h[2]\, \cancel{x[-1]}^{\,0} + \ldots$

$1 = 0 + h[1] (\frac{1}{2})^0 = h[1] \quad \Rightarrow h[1] = 1$

$y[2] = h[0]\, x[2] + h[1]\, x[1] + h[2]\, x[2]$

$0 = 0 + 1 (\frac{1}{2})^1 + h[2] (\frac{1}{2})^0$

$0 = \frac{1}{2} + h[2] \quad \Longrightarrow \quad h[2] = -\frac{1}{2}$

$y[3] = h[0]\, x[3] + h[1]\, x[2] + h[2]\, x[1] + h[3]\, x[0].$

$0 = 0 + 1 (\frac{1}{2})^2 - \frac{1}{2} (\frac{1}{2})^1 + h[3] (\frac{1}{2})^0$

$0 = 0 + \frac{1}{4} - \frac{1}{4} + h[3] \Rightarrow h[3] = 0$

Similarly for $n > 3$

$\therefore h[n] = \delta[n-1] - \frac{1}{2} \delta[n-2]$

Problem 7.3:

The MATLAB program has two filters that are added together, and then filtered again

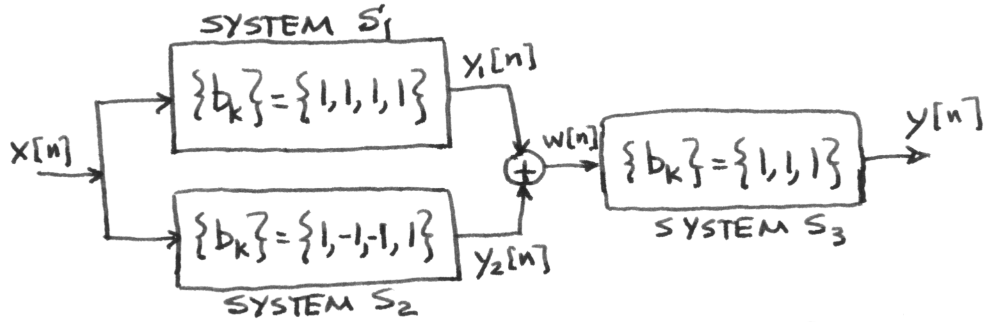
$$y_1[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

$$y_2[n] = x[n] - x[n-1] - x[n-2] + x[n-3]$$

$$w[n] = y_1[n] + y_2[n]$$

$$y[n] = w[n] + w[n-1] + w[n-2]$$

(a)



$$S_1: \; h_1[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$S_2: \; h_2[n] = \delta[n] - \delta[n-1] - \delta[n-2] + \delta[n-3]$$

$$S_3: \; h_3[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

(b) When $x[n] = \delta[n]$, $w[n] = h_1[n] + h_2[n]$

$$= 2\delta[n] + 2\delta[n-3]$$

Then $y[n] = h_3[n] * w[n]$

$$= 2\delta[n] + 2\delta[n-1] + 2\delta[n-2] + 2\delta[n-3] + 2\delta[n-4] + 2\delta[n-5]$$

The overall difference equation is obtained by noting that the filter coeffs are equal to the impulse response values: $b_k = h[n]\big|_{n=k}$

$$y[n] = 2x[n] + 2x[n-1] + 2x[n-2] + 2x[n-3] + 2x[n-4] + 2x[n-5]$$

Problem 7.4

a) $y[n] = x[n-3] - 2x[n] + x[n+3]$

This system is described by a linear, constant coefficient difference equation, so it is Linear and time-invariant. Because of the last term, however, this system is not causal. Note that if $x[n] = \delta[n]$ then $y[n] = \delta[n+3]$, i.e. the system responds before it is excited.

b) $y[n] = (x[3-n])^2$

This system is clearly non-linear. To show this, note for an input $x[n]$ the output is $y[n] = (x[3-n])^2$ and with $x'[n] = a\,x[n]$ the output is

$$y'[n] = (x'[3-n])^2 = (a\,x[3-n])^2 = a^2 (x[3-n])^2$$

$$= a^2 y[n] \neq a\,y[n]$$

This system is also time-varying. To see this, note that with an input $x[n] = \delta[n]$ the output is $y[n] = \delta[3-n]$ whereas if $x'[n] = \delta[n-1]$, i.e. delay $x[n]$ by one, then

$$y'[n] = (x'[3-n])^2 = \delta[2-n] \neq y[n-1]$$

Finally, this system is non-causal since

$$x[n] = \delta[n-3] \longrightarrow y[n] = \delta[n]$$

so we have a response before there is an input.

c) $y[n] = x[n] \cos(0.3\pi n)$

This system is linear since, if $x[n] = a x_1[n] + b x_2[n]$ then the response is

$$y[n] = \{ a x_1[n] + b x_2[n] \} \cos(0.3\pi n)$$

$$= a \{ x_1[n] \cos(0.3\pi n) \} + b \{ x_2[n] \cos(0.3\pi n) \}$$

$$= a y_1[n] + b y_2[n].$$

Because the output is formed by multiplying $x[n]$ by a function of time, $\cos(0.3\pi n)$, this system is time-varying (think of this system as one that has a time-varying gain). The show that it is time-varying compare the responses to $x_1(n) = \delta(n)$ and $x_2(n) = \delta[n-1]$. The response to the first is $y_1[n] = \delta[n]$ whereas for the second it is $y_2[n] = \cos(0.3\pi)\,\delta[n-1] \neq y_1[n-1]$

Finally, this system is causal because the output at time n depends only on the input at time n (this is called a memoryless system).

Problem 7.5: In this problem we are given the response of a linear time-invariant system to the input $x_1[n] = \delta[n+3]$. From this, we want to find the response to

$$x_2[n] = \delta[n+1] + 2\delta[n-2]$$

Note that $x_1[n-3] = \delta[n]$. Therefore, using time-invariance we know that the unit sample response is $h[n] = y_1[n-3]$, or

$$h[n] = \delta[n-3] + 2\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 1.5\delta[n-7]$$

Thus, $$y_2[n] = h[n+1] + 2h[n-2]$$

which may be found using tabulation as shown below:

n:	0	1	2	3	4	5	6	7	8	9	10
$h[n+1]$			1	2	4	3	1.5				
$2h[n-2]$						2	4	8	6	3	
$y_2[n]$			1	2	4	5	5.5	8	6	3	

Yes, this system is causal.

Problem 7.6. $y[n] = \sum_{k=0}^{M} b_k x[n-k]$

a) Filter length $= M+1$

b) If $x[n]$ is non-zero only for $8 \le n \le 17$ and $M=7$, then

i) The first non-zero value of $y[n]$ will occur at index $n=8$ (assuming $b_0 \neq 0$).

ii) The index of the last non-zero value of $y[n]$ will occur at index $n = 17+7 = 24$

iii) The total length of the input sequence is $L = 17-8+1 = 10$.

c) If the input sequence is non-zero only for $N_1 \le n \le N_2$ then the length of the input sequence is $L = N_2 - N_1 + 1$.

d) The input in (b) has the form

$$x[n] = \sum_{k=8}^{17} \alpha_k \delta[n-k]$$

Therefore, the output is

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \left[\sum_{k=8}^{17} \alpha_k \delta[n-k] \right] * h[n] \\ &= \sum_{k=8}^{17} \alpha_k h[n-k] \end{aligned}$$

which is a weighted sum of delayed impulse responses

Since the first non-zero value of $h[n]$ occurs at $n=0$, then the first non-zero value in $y[n]$ occurs at $n=8$, so

$$N_3 = 8$$

And, since the last non-zero value of $h[n]$ occurs at index $n = 17 + 7 = 24$ (recall that $M=7$), so

$$N_4 = 24$$

e) The length of the output sequence is

$$\underbrace{(17-8+1)}_{\text{length of } x[n]} + \underbrace{(7+1)}_{\text{length of } h[n]} - 1 = 17$$

or, using the results of part (d)

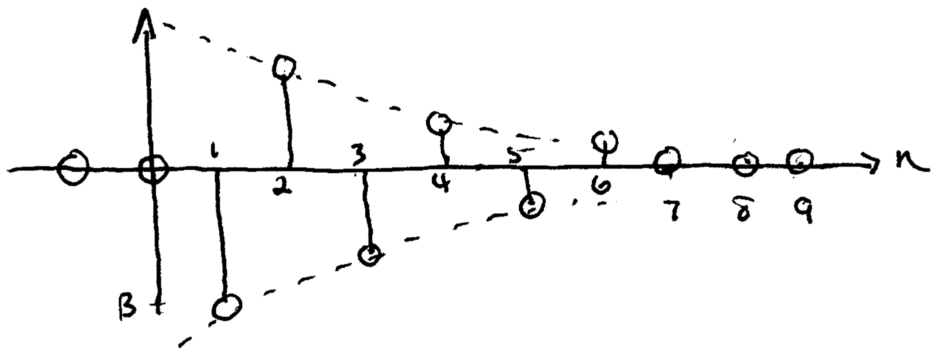
$$N_4 - N_3 + 1 = 17$$

Problem 7.7

a) We are given $h_2[n]$ to be

$$h_2[n] = \sum_{k=1}^{6} (-\beta)^k \delta[n-k]$$

This is plotted below assuming β is positive and less than one, $0 < \beta < 1$



b) To find the impulse response of the cascade, we first note that $h_1[n] = \delta[n-2] + \beta\,\delta[n-3]$. Now,

$$h[n] = h_1[n] * h_2[n]$$

$$= \left\{ \delta[n-2] + \beta\,\delta[n-3] \right\} * h_2[n]$$

$$= h_2[n-2] + \beta\, h_2[n-3]$$

$$= -\beta\, \delta[n-3] + \beta^7 \delta[n-9]$$

There are several ways to show the last equality. You may show it graphically, using a table, or mathematically. For example, to show it mathematically we have:

$$h[n] = h_2[n-2] + \beta h_2[n-3]$$

$$= \sum_{k=1}^{6} (-\beta)^k \delta[n-2-k] + \beta \sum_{k=1}^{6} (-\beta)^k \delta[n-3-k]$$

$$= \sum_{k=1}^{6} (-\beta)^k \delta[n-2-k] + \beta \sum_{k=2}^{7} (-\beta)^{k-1} \delta[n-2-k]$$

$$= \sum_{k=1}^{6} (-\beta)^k \delta[n-2-k] - \sum_{k=2}^{7} (-\beta)^k \delta[n-2-k]$$

$$= (-\beta) \delta[n-3] - (-\beta)^7 \delta[n-9]$$

$$= -\beta \delta[n-3] + \beta^7 \delta[n-9]$$

phew!

c) with $h[n] = -\beta \delta[n-3] + \beta^7 \delta[n-9]$

$$y[n] = -\beta x[n-3] + \beta^7 x[n-9]$$

If $\beta = 0.75$ then

$$y[n] = -0.75\, x[n-3] + (0.75)^7 x[n-9]$$