PROBLEM 6.1*:

A linear time-invariant discrete-time system is described by the difference equation

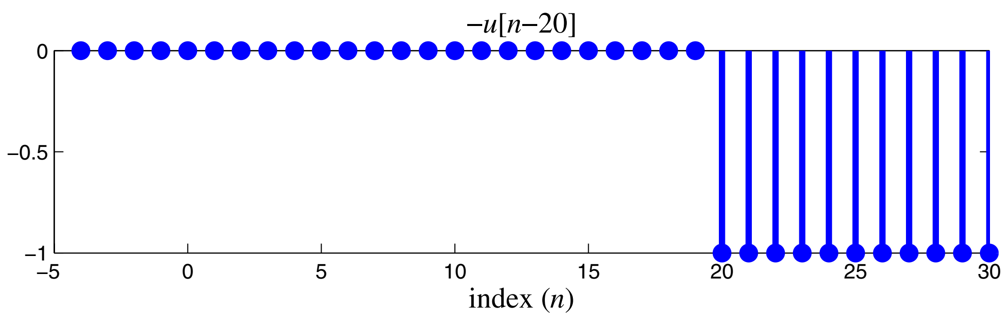
$$y[n] = -x[n] - 2x[n-1] - 3x[n-2] - 4x[n-3]$$

(a) Replace $x[n]$ with $\delta[n]$ to get the impulse response $h[n] = -\delta[n] - 2\delta[n-1] - 3\delta[n-2] - 4\delta[n-3]$

(b) The filter coefficients b_k can be read from the difference equation: $\{b_k\} = \{-1,\ -2,\ -3\ -4\}$

(c) The *order* of the filter (M) is derived from the largest delay term, so $-4x[n-3] \;\Rightarrow\; M = 3$.
The *length* of the filter (L) is $L = M + 1 = 4$; it is also the number of filter coefficients.

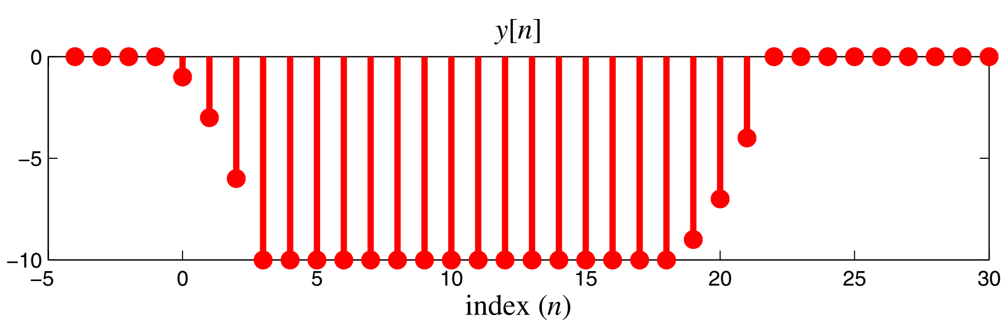
(d) Here is a plot of the shifted unit-step signal $s[n] = -u[n-20]$ which is zero up to $n = 20$.



(e) The convolution table below computes the output due to the input $x[n] = u[n-1] - u[n-20]$

n	$n \le 0$	0	1	2	3	4	5	…	17	18	19	20	21	22	23	$n > 23$
$x[n]$	0	0	1	1	1	1	1	…	1	1	1	0	0	0	0	0
$h[n]$	0	−1	−2	−3	−4											
$h[0]x[n]$	0	0	−1	−1	−1	−1	−1	…	−1	−1	−1	0	0	0	0	0
$h[1]x[n-1]$	0	0	0	−2	−2	−2	−2	…	−2	−2	−2	−2	0	0	0	0
$h[2]x[n-2]$	0	0	0	0	−3	−3	−3	…	−3	−3	−3	−3	−3	0	0	0
$h[3]x[n-3]$	0	0	0	0	0	−4	−4	…	−4	−4	−4	−4	−4	−4	0	0
$y[n]$	0	0	−1	−3	−6	−10	−10	…	−10	−10	−10	−9	−7	−4	0	0

Here is a plot of $y[n]$ over its nonzero range:



PROBLEM 6.2*:
For each of the following systems, determine if they are (1) linear; (2) time-invariant; (3) causal.

(a) $y[n] = x[n]\,x[n-1]$ (Multiplier)

(1) ***Linear: No*** [*Need a counter-example*]
Consider the test signal $x_1[n] = u[n]$ for which the output is

$$y_1[n] = u[n]u[n-1] = u[n-1]$$

Then examine a second input signal $x_2[n] = 2u[n]$ which gives the output

$$y_2[n] = 2u[n]2u[n-1] = 4u[n-1]$$

Although $x_2[n] = 2x_1[n]$, it is clear that $y_2[n] \neq 2y_1[n]$, so the scaling property of linearity does not hold.

(2) ***Time-Invariant: Yes***
To prove this in general, let $v[n] = x[n-n_d]$ and then make $v[n]$ the input to the system. The corresponding output would be $w[n] = v[n]v[n-1]$ which becomes $w[n] = x[n-n_d]x[n-1-n_d]$. We must compare this result to a shifted version of the output, i.e., compare to $y[n-n_d]$. From the system definition, we get $y[n-n_d] = x[n-n_d]x[n-n_d-1]$. Since $w[n] = y[n-n_d]$ for any integer n_d, time invariance is true.

(3) ***Causal: Yes***
Since the output is computed from the present input, $x[n]$ and the previous input, $x[n-1]$, no inputs from the future are required to compute $y[n]$.

(b) $y[n] = e^{jx[n-3]}$ (Complex Exponential)

(1) ***Linear: No*** [*Need a counter-example*]
Consider the test signal $x_1[n] = \pi\delta[n]$ for which the output is

$$y_1[n] = e^{j\pi\delta[n-3]} = 1 + e^{j\pi}\delta[n-3] = 1 - \delta[n-3]$$

Then examine a second input signal $x_2[n] = 2\pi\delta[n]$ which gives the output

$$y_2[n] = e^{j2\pi\delta[n-3]} = 1 + e^{j2\pi}\delta[n-3] = 1 + \delta[n-3]$$

Although $x_2[n] = 2x_1[n]$, it is clear that $y_2[n] \neq 2y_1[n]$, so the scaling property of linearity does not hold.

(2) ***Time-Invariant: Yes***
To prove this in general, let $v[n] = x[n-n_d]$ and then make $v[n]$ the input to the system. The corresponding output would be $w[n] = e^{jv[n-3]}$ which becomes $w[n] = e^{jx[n-3-n_d]}$. We must compare this result to a shifted version of the output, i.e., compare to $y[n-n_d]$. From the system definition, we get $y[n-n_d] = e^{jx[n-n_d-3]}$. Since $w[n] = y[n-n_d]$ for any integer n_d, time invariance is true.

(3) ***Causal: Yes***
Since the output is computed from a previous input, $x[n-3]$, no future inputs are required to compute $y[n]$.

Note: there is an argument that says the system is NOT causal, if you say that causality means that the output does not start before the input. Consider the input signal $x_1[n] = \pi\delta[n]$ for which the output is $y_1[n] = 1 - \delta[n-3]$. Since $x_1[n]$ starts at $n = 0$, but $y_1[n]$ is already nonzero for $n < 0$, it might be argued that the system in not causal. However, this is not the preferred interpretation.

(c) $y[n] = x[n^2 - 1]$ (Time Distortion)

(1) ***Linear: Yes***

Since the system operates on the n variable, the superposition and scaling properties will be true. Thus, we assume that $y_1[n] = x_1[n^2 - 1]$ and $y_2[n] = x_2[n^2 - 1]$. If we form $x_3[n] = \alpha x_1[n] + \beta x_2[n]$, then the corresponding output will be

$$y_3[n] = \alpha x_1[n^2 - 1] + \beta x_2[n^2 - 1] = \alpha y_1[n] + \beta y_2[n]$$

(2) ***Time-Invariant: No*** [*Need a counter-example*]

Consider the test signal $x_1[n] = \delta[n]$ for which the output is

$$y_1[n] = \delta[n^2 - 1] = \delta[n+1] + \delta[n-1]$$

And compare to the shifted input $x_2[n] = x_1[n+1] = \delta[n+1]$ where the output is

$$y_1[n] = \delta[(n^2 - 1) + 1] = \delta[n^2] = \delta[n]$$

Since $y_2[n] \neq y_1[n+1]$, the system is *not* time-invariant.

(3) ***Causal: No*** [*Need a counter-example*]

Consider the test signal $x[n] = \delta[n]$ for which the output is $y[n] = \delta[n+1] + \delta[n-1]$. Although the input starts at $n = 0$, the output starts at $n = -1$.

PROBLEM 6.3*:

(a) $x[n] = u[n]$ and $y[n] = u[n-1]$

We need a "delay by one".

$$\Rightarrow h[n] = \delta[n-1]$$

(b) $x[n] = u[n]$ and $y[n] = \delta[n]$

Since $u[n]$ jumps from 0 to 1 at $n=0$, we need a filter that detects jumps. This can be done with a first-difference filter.

$$h[n] = \delta[n] - \delta[n-1]$$

(c) $x[n] = \left(\frac{1}{2}\right)^n u[n]$ and $y[n] = \delta[n-1]$

Use the convolution sum to write linear equations:

$$y[n] = \sum_{k=0}^{M} h[k]\,x[n-k].$$

$$y[0] = h[0]\,x[0] + h[1]\,x[-1] + \ldots$$

Note: $x[n]=0$ for $n<0$

$$0 = h[0]\left(\tfrac{1}{2}\right)^0 = h[0] \qquad \Rightarrow h[0] = 0$$

$$y[1] = h[0]\,x[1] + h[1]\,x[0] + \cancelto{0}{h[2]\,x[-1]} + \ldots$$

$$1 = 0 + h[1]\left(\tfrac{1}{2}\right)^0 = h[1] \quad \Rightarrow h[1] = 1$$

$$y[2] = h[0]\,x[2] + h[1]\,x[1] + h[2]\,x[2]$$

$$0 = 0 + 1\left(\tfrac{1}{2}\right)^1 + h[2]\left(\tfrac{1}{2}\right)^0$$

$$0 = \tfrac{1}{2} + h[2] \quad \Longrightarrow \quad h[2] = -\tfrac{1}{2}$$

$$y[3] = h[0]\,x[3] + h[1]\,x[2] + h[2]\,x[1] + h[3]\,x[0].$$

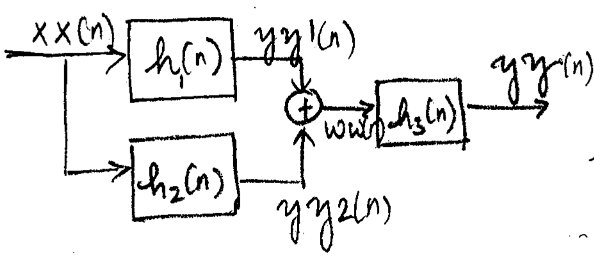
$$0 = 0 + 1\left(\tfrac{1}{2}\right)^2 - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^1 + h[3]\left(\tfrac{1}{2}\right)^0$$

$$0 = 0 + \tfrac{1}{4} - \tfrac{1}{4} + h[3] \Rightarrow h[3] = 0$$

Similarly for $n > 3$

$$\therefore\ h[n] = \delta[n-1] - \tfrac{1}{2}\,\delta[n-2]$$

6.4



$$h_1(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{ow} \end{cases}$$

$$h_2(n) = \begin{cases} 1 & n = 0, 3 \\ -1 & n = 1, 2 \\ 0 & \text{ow} \end{cases}$$

$$h_3(n) = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{ow} \end{cases}$$

$yy1 = xx(n) * h_1(n)$

$yy2 = xx(n) * h_2(n)$

$ww(n) = yy_1(n) + yy2(n)$

$$\begin{aligned} yy(n) &= h_3(n) * ww(n) \\ &= [xx(n) * h_1(n) + xx(n) * h_2(n)] * h_3(n) \\ &= xx(n) * h_1(n) * h_3(n) + xx(n) * h_2(n)\, h_3(n) \\ &= xx(n) * [h_1(n) * h_3(n) + h_2(n)\, h_3(n)], \end{aligned}$$

$h_4(n) = h_1(n) * h_3(n) + h_2(n) * h_3(n)$

$yy(n) = xx(n) * h_4(n)$

$$h_1[n] + h_2(n) = 2\delta(n) + 2\delta(n-3)$$

$$h_3[n] = \delta(n) + \delta(n-1) + \delta(n-2)$$

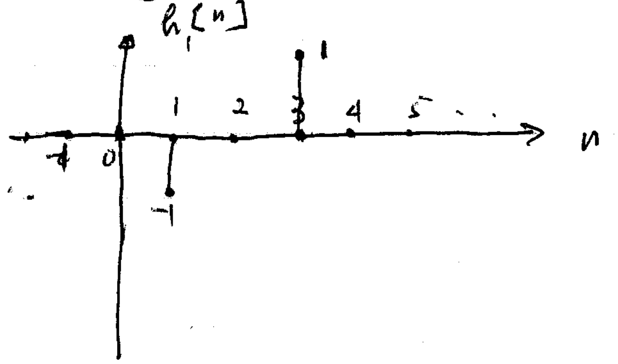
$$h_4(n) = 2[\delta(n) + \delta(n-3) + \delta(n-1) + \delta(n-4) + \delta(n-2) + \delta(n-5)]$$

so $y(n) = 2[x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4) + x(n-5)]$

$y(n)$ is 12 times a 6-point averager

sol. Prob. 6.5 :

(a) $h_1[n] = \delta(n-3) - \delta(n-1)$



(b) $h[n] = h_1[n] * h_2[n]$

$h_2[n] = \delta(n-1) - \delta(n)$

Conv. with $\delta(n-1)$: one delay

" " $\delta(n)$: no delay

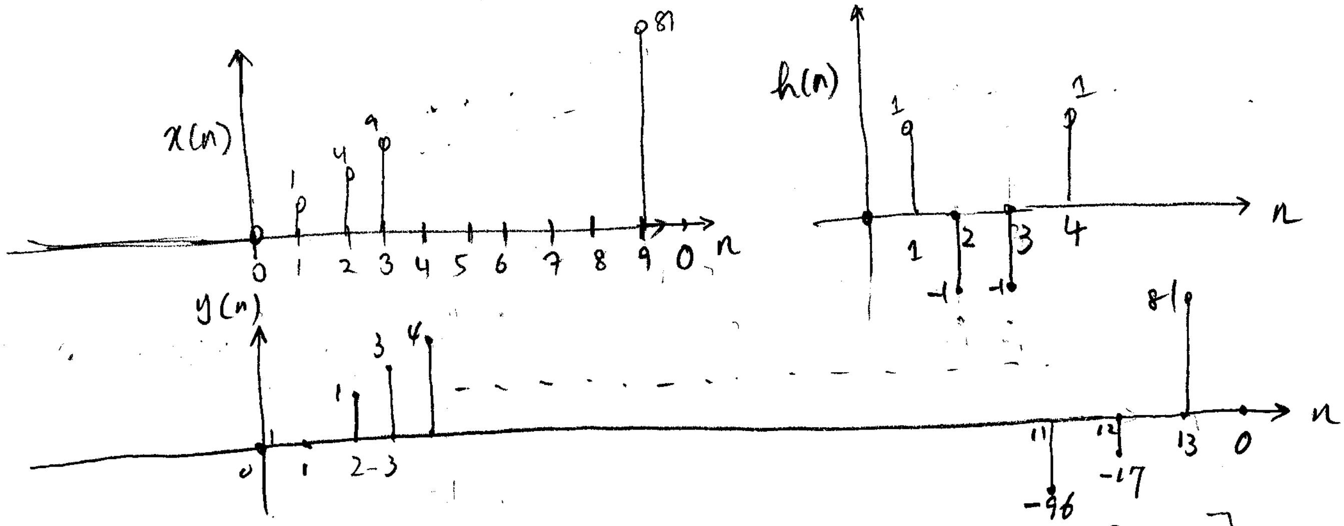
$$h[n] = [\delta(n-3) - \delta(n-1)] * \delta(n-1)$$
$$- [\delta(n-3) - \delta(n-1)] * \delta(n)$$
$$= \delta(n-4) - \delta(n-2)$$
$$- \delta(n-3) + \delta(n-1)$$

(Can verify this using Table)

(c) Convolve with $\delta(n-n_0)$: n_0 delays

$$y(n) = x(n) * h(n)$$
$$= x(n-4) - x(n-3) - x(n-2) + x(n-1)$$
$$= (n-4)^2[u(n-4) - u(n-14)] - (n-3)^2[u(n-3) - u(n-13)]$$
$$- (n-2)^2[u(n-2) - u(n-12)] + (n-1)^2[u(n-1) - u(n-11)]$$

$[u(n) - u(n-10)]$



$y[n] = [0, 0, 1, 3, 4, 4, 4, 4, 4, 4, 4, -96, -17, 81]$

$$y(n) = \begin{cases} 0 & n \le 1,\ n \ge 14 \\ \ne 0 & 2 \le n \le 13 \end{cases}$$

6.6 $\quad y(n) = \sum_{k=0}^{M} b_k x(n-k)$

a) filter length $M+1$

b) $x(n) \ne 0$ for $10 \le n \le 20$ & $M = 9$

$$y(n) = \sum_{k=0}^{9} b_k x(n-k)$$

first non-zero $y[10] = b_0 x(10) + b_1 x(9) + \dots + b_9 x(1)$

Index of last non-zero output sequence $y[29]$

(c) $x(n)$ is non zero for $N_1 \le n \le N_2$

length of I/P sequence $N_2 - N_1 + 1$

(d) $y(n)$ is non-zero for $N_3 \le n \le N_4$ $\quad N_3 = 10$

$N_4 = 20 + M = 29$

(e) 20 samples $= N_4 - N_3 + 1$