

GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 2003
Problem Set #12

Assigned: 17-Nov-03
Due Date: 5-Dec-03

Reading: In *Signal Processing First*, Sections 6 through 11 of Chapter 11 on the *Continuous-Time Fourier Transform* and Chapter 12 on *Filtering, Modulation, and Sampling*

The due date for this homework will be Friday, Dec. 5, in lecture, for everyone. You are welcome to turn it in early during your recitation that week if you'd prefer; we'd like to encourage people to start on it early and get it done and out of the way.

⇒ Please check the "Bulletin Board" often. All official course announcements are posted there.

ALL of the **STARRED** problems will have to be turned in for grading.

A solution will be posted to the web.

Your homework is due in recitation at the beginning of class. After the beginning of your assigned recitation time, the homework is considered late and will be given a zero.

PROBLEM 12.1:

Try parts (b) and (c) of P-11.6 on p. 342-343. Notice I'm particularly drawing your attention to those two parts. That might be a clue to you that you might see similar questions again sometime, for instance, during exam week.

PROBLEM 12.2*:

Do P-11.8 on p. 343 of *Signal Processing First*.

PROBLEM 12.3:

Try P-11.15 on *Signal Processing First*. Manipulate your answers to parts (a) and (b) to make sure they are really the same thing. This problem illustrates cool ways of using a lot of cool tricks. We assign it often as homework; you will probably be able to uncover a solution to it in the WORD section of Web CT if you poke around a bit.

PROBLEM 12.4*:

The homework problems that one gets in engineering and math classes often feel unrelated to anything in the real world. To try to make this material more “real,” the exposition in this problem and the next problem “dresses up” some standard ECE2025 questions with some background information about where these things actually show up in real life! (If you don’t care about the background, you need not pay close attention to it. You can just work the problems, but hopefully you’ll find the background interesting.)

- (a) The “ramp filter” is often used in medical imaging applications such as X-ray computer-aided tomography. It has a frequency response given by

$$H(j\omega) = \begin{cases} j\omega & \text{for } |\omega| < \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Find the impulse response, $h(t)$, of this filter. (You may need to go back to your old calculus text if you’ve forgotten how to take derivatives of quotients.) Also make a plot of the impulse response (this is pretty complicated, so you should feel free to use MATLAB or one of those fancy graphing calculators to help you make the plot.) If you’ve ever had a CAT scan done, you’ve run across this filter in practice!

A hint on making the plot: We’re interested in the overall shape of the curve; you can pick whatever ω_0 you find the most interesting. I found that the following bit of MATLAB code made a nice plot. You should feel free to use it:

```
omega_0 = 2*pi;
period = 2*pi/omega;
t = -5*period:period/100:5*period;
h = [fill in code that goes here]
h([t == 0]) = [fill in value that goes here]
plot(t,h);
```

You may ask yourself what the $h([t == 0]) =$ business is all about. Well, it turns out that $h(t)$ is indeterminate at $t = 0$, i.e. we wind up dividing zero by zero. To find a meaningful value for $h(0)$, use L’Hopital’s rule. Alternatively, you can probably guess what $h(0)$ should be by looking at $h(t)$ for t near zero.

- (b) Chemists using *nuclear magnetic resonance* and radiologists using *magnetic resonance imaging*¹ spend a lot of time thinking about *free induction decay signals*, which take the form of a decaying sinusoid, for instance:

$$x(t) = A \exp\left(-\frac{t}{T_2}\right) \sin(\omega_0 t) u(t)$$

¹To be consistent, what we know now as MRI should be called NMRI. However, many of these developments occurred during the height of the cold war, and using the word “nuclear” was considered unwise from a marketing perspective. Hence, the “N” got dropped.

The nonnegative constant T_2 is called the *spin-spin relaxation time*. Different kinds of body tissue have different T_2 times; for instance, by determining T_2 times, doctors can identify and locate some kinds of cancer. ω_0 is called the resonant frequency; in an MRI scheme, this tells you what part of the body the signal is coming from. A is called the *spin density*, which also depends on tissue type.

Again, if you don't find anything in the previous paragraph at all interesting, don't worry about it; all I want you to do is find $X(j\omega)$, the Fourier transform of $x(t)$, in terms of the constants A , T_2 , and ω_0 .

- (c) OK, I couldn't come up with a good motivating application for this last subproblem, but it's still a good problem: find the Fourier transform of $x(t) = e^{-8t}u(t) - e^{-8t}u(t - 2)$. (Hint: first rearrange it into the form $x(t) = e^{-8t}u(t) - \beta e^{-8(t-2)}u(t - 2)$.) Also make a plot of $x(t)$.

PROBLEM 12.5*:

- (a) On p. 325 of
- Signal Processing First*
- , the authors develop a Fourier transform pair

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2 + \omega^2} \quad (1)$$

You may have noticed that Fourier transform pairs have a kind of symmetry. For instance, rectangular pulses in the time domain transform to sincs in the frequency domain, and sincs in the time domain transform to rectangular windows in the frequency domain. Similarly, (1) has a related Fourier transform pair:

$$\frac{a}{\pi(a^2 + t^2)} \xleftrightarrow{\mathcal{F}} e^{-a|\omega|}$$

- (i) Suppose
- $x(t) = a/[\pi(a^2 + t^2)]$
- and
- $h(t) = b/[\pi(b^2 + t^2)]$
- . Compute the convolution
- $y(t) = x(t) * h(t)$
- . Theoretically, you could find the result by directly cranking on the convolution integral:

$$y(t) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{a}{\pi(a^2 + \tau^2)} \frac{b}{\pi(b^2 + (t - \tau)^2)} d\tau$$

However, that way lies *madness*. There is an easier path!

- (ii) When you convolve two similar functions, you usually don't get a function that looks anything like the functions you're convolving. For instance, if you convolve two rectangular pulses, you get a triangle or a trapezoid. However, there are a few families of functions – for instance, consider the sinc function example in Section 11-6.3.1 – that give you a member of the same family if you convolve two functions from that family. Such families of functions are relatively rare and usually have quite interesting properties in many applications, particularly probability theory.

Looking at your answer to part (a), does it seem very different in form than $x(t)$ or $y(t)$? Or does it look similar?

- (b) If you shine light with wavelength
- λ
- through a small slit with width
- d
- , the intensity of the diffracted pattern is well-approximated by the function

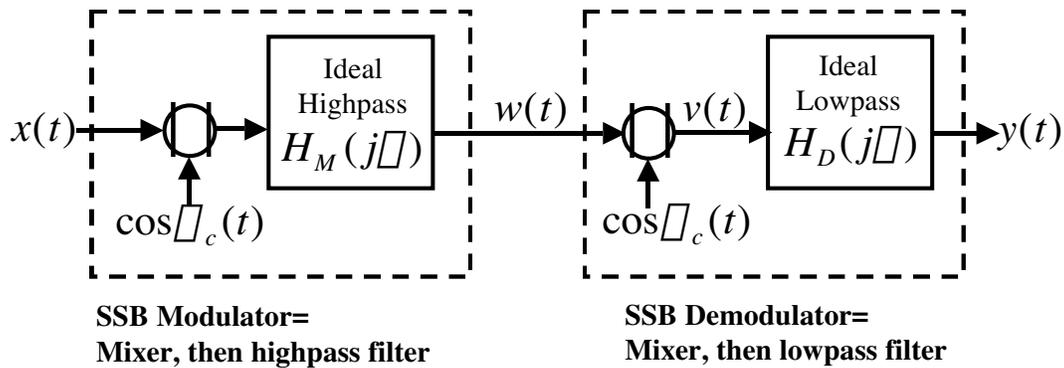
$$x(t) = \left[\frac{\sin\left(\frac{\pi d t}{\lambda}\right)}{\frac{\pi d t}{\lambda}} \right]^2$$

(Normally, we wouldn't use t , since here it represents position instead of time. However, we'll stick with t since that's what you're used to seeing.)

Find the Fourier transform $X(j\omega)$. (Hint: you can use reasoning similar to that found in Sec. 11-6.3.2 on p. 329 of the text; this problem is the dual of that example.)

PROBLEM 12.6*:

In lecture, we explored a communication strategy called *double-sideband amplitude modulation*. In this problem, we will explore another strategy called *single-sideband modulation* (SSB). Block diagrams of an SSB Modulator and an SSB Demodulator are given below.



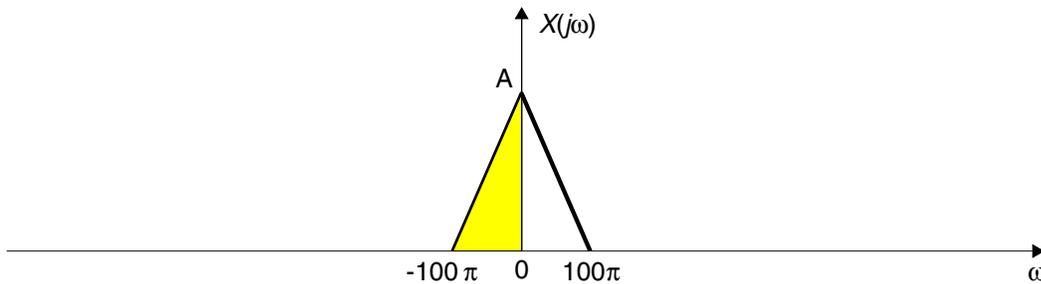
H_M is a *highpass filter*, here specified by:

$$H_M(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq 500\pi \\ 0 & \text{otherwise} \end{cases}$$

H_D is a *lowpass filter*, here specified by:

$$H_D(j\omega) = B[u(\omega + 200\pi) - u(\omega - 200\pi)]$$

Notice this looks very similar to the double-sideband AM communication system you explored in lab; the only change is the addition of the highpass filter H_M to the modulator. Suppose that $\omega_c = 500\pi$, and the input signal $x(t)$ has this “typical” Fourier transform $X(j\omega)$:



We will now trace the input signal through the modulation and demodulation stages to the output by analyzing it in the Fourier domain. In your sketches, be sure to **label the amplitudes and correctly note what parts are “shaded” and what parts are “unshaded.”** (Keeping careful track of the amplitudes will be important to part (c).) You will find it easiest to think *graphically*; you will not need to write any complicated equations.

- (a) Sketch $W(j\omega)$, the Fourier transform of $w(t)$.
- (b) Sketch $V(j\omega)$, the Fourier transform of $v(t)$.
- (c) How should B , the gain of the lowpass filter in the demodulator, be chosen so that the input exactly equals the input, i.e. $y(t) = x(t)$? (Hint: it may help to sketch $Y(j\omega)$.)
- (d) SSB modulation looks like it is probably more complicated to implement than DSBAM. What aspect of SSB may make it preferable to using DSBAM? (This is a thinking question.)

PROBLEM 12.7:

Try Exercise 12.8 on p. 362 of *Signal Processing First*. You will find the solution on the CD-ROM which comes with the textbook, or the website which mirrors the CD-ROM materials. (This exercise is extremely instructive, and explains why real AM radios don't work the way we presented in class! You can see Sec. 12-2.2 of the text for an explanation of how the real ones work – and no, Sec. 12-2.2 will not be on the final. It's just good to know, you know?)

PROBLEM 12.8:

Stare at Equation 12.49 on p. 373 of *Signal Processing First*. Contemplate it deeply. Think to yourself, "Aaron is very likely to put a question on the final exam that will test whether I understand what this Equation 12.49 *does*."

PROBLEM 12.9*:

Do problem P-12.12 on p. 386 of *Signal Processing First*.