Georgia Institute of Technology

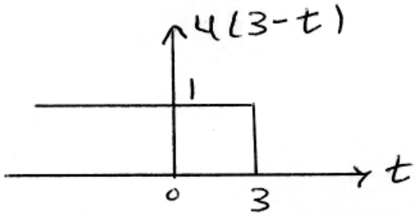
School of ECE

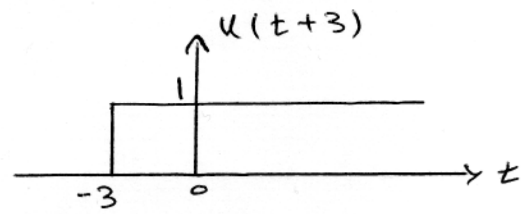
ECE 2025

HW 11 Solutions

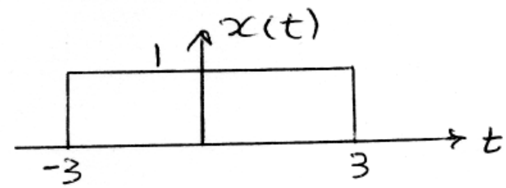
11.1

(a)





$x(t) = u(t+3)\cdot u(3-t) \longrightarrow$



$T_0/2 = 3 \rightarrow X(j\omega) = \dfrac{\sin(3\omega)}{\omega/2}$

(b) Note

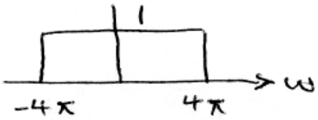
$$\underset{t\text{-domain}}{\sin 4\pi t} \longleftrightarrow \underset{\omega\text{-domain}}{\frac{\pi}{j}\delta(\omega-4\pi) - \frac{\pi}{j}\delta(\omega+4\pi)}$$

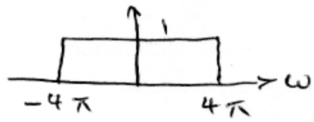
Convolution property of the Fourier Transform:

$$X(j\omega) = \frac{1}{2\pi}\left\{\frac{\pi}{j}\delta(\omega-4\pi) - \frac{\pi}{j}\delta(\omega+4\pi)\right\} * \left\{\frac{\pi}{j}\delta(\omega-50\pi) - \frac{\pi}{j}\delta(\omega+50\pi)\right\}$$

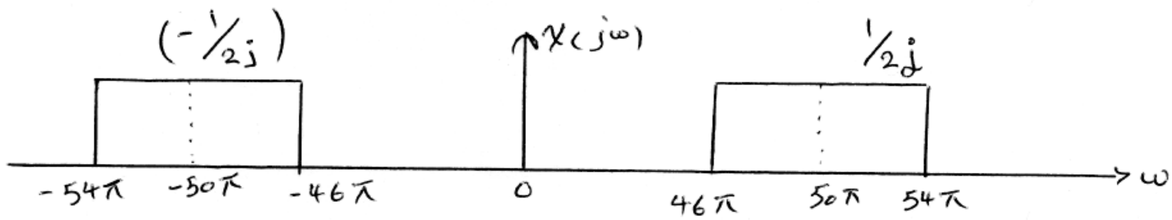
$$X(j\omega) = \frac{\pi}{2}\delta(\omega-46\pi) + \frac{\pi}{2}\delta(\omega+46\pi) - \frac{\pi}{2}\delta(\omega-54\pi) - \frac{\pi}{2}\delta(\omega+54\pi)$$

11.1

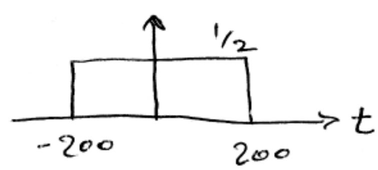
(c) Note $\dfrac{\sin 4\pi t}{\pi t} \longleftrightarrow$ 

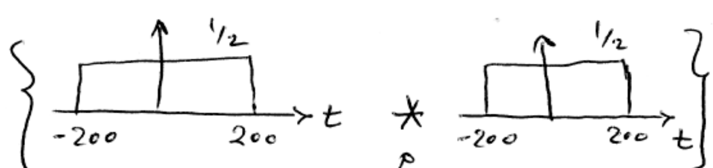
$X(j\omega) = \dfrac{1}{2\pi}\Big\{$  $\Big\} \underset{\text{convolution}}{*} \Big\{ \dfrac{\pi}{j}\,\delta(\omega - 50\pi) - \dfrac{\pi}{j}\,\delta(\omega + 50\pi) \Big\}$

Thus

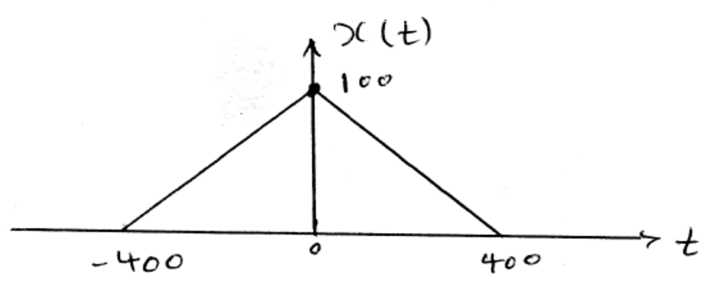


$$X(j\omega) = \begin{cases} \dfrac{1}{2j} & 46\pi \le \omega \le 54\pi \\ -\frac{1}{2j} & -54\pi \le \omega \le -46\pi \\ 0 & \text{else} \end{cases}$$

(d) Note: $\dfrac{1}{2}\,\dfrac{\sin(200\omega)}{\omega/2} \longleftrightarrow$ 

$\dfrac{\sin^2(200\omega)}{\omega^2} \longleftrightarrow$ 

Thus



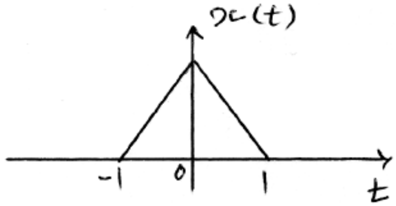
(e) $\cos\omega \longleftrightarrow \dfrac{1}{2}\{\delta(t-1) + \delta(t+1)\}$

$\cos^2\omega \longleftrightarrow \dfrac{1}{2}\{\delta(t-1) + \delta(t+1)\} * \dfrac{1}{2}\{\delta(t-1) + \delta(t+1)\}$

$x(t) = \dfrac{1}{4}\{\delta(t-2) + \delta(t+2) + 2\delta(t)\}$

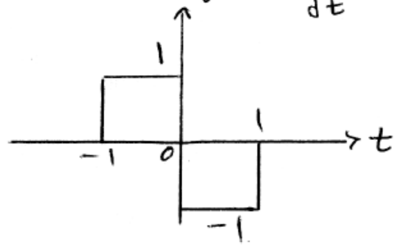
11.2

(a)

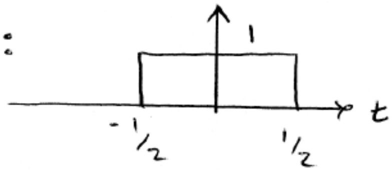


$\Rightarrow$

$y(t) = \frac{d}{dt}x(t)$

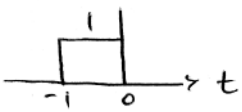


since:

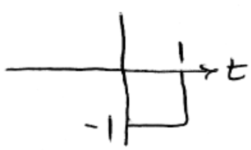


$$\longleftrightarrow \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}$$

then



$$\longleftrightarrow e^{j\frac{\omega}{2}}\frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}$$



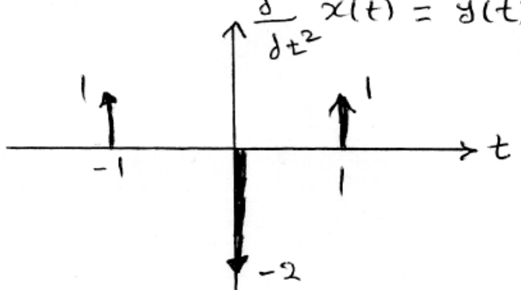
$$\longleftrightarrow (-1)e^{-j\frac{\omega}{2}}\frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}$$

Thus: $$Y(j\omega) = \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) = \frac{4j}{\omega}\sin^2\left(\frac{\omega}{2}\right)$$

$$Y(j\omega) = j\omega\, X(j\omega) \Longrightarrow X(j\omega) = \frac{4}{\omega^2}\sin^2\frac{\omega}{2} = \left(\frac{\sin\frac{\omega}{2}}{\omega/2}\right)^2$$

(b)

$\frac{d^2}{dt^2}x(t) = y(t)$



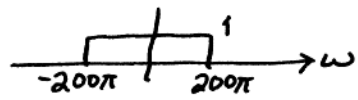
$$Y(j\omega) = e^{j\omega} + e^{-j\omega} - 2 = 2\cos\omega - 2 = -4\sin^2\left(\frac{\omega}{2}\right)$$

$$X(j\omega) = \left(\frac{1}{j\omega}\right)^2 Y(j\omega) = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right)$$

Prob 11.3

(a) Use derivative property: $\frac{d}{dt}x(t) \longrightarrow j\omega X(j\omega)$

F.T. of $\frac{\sin(200\pi t)}{\pi t}$ is a rectangle

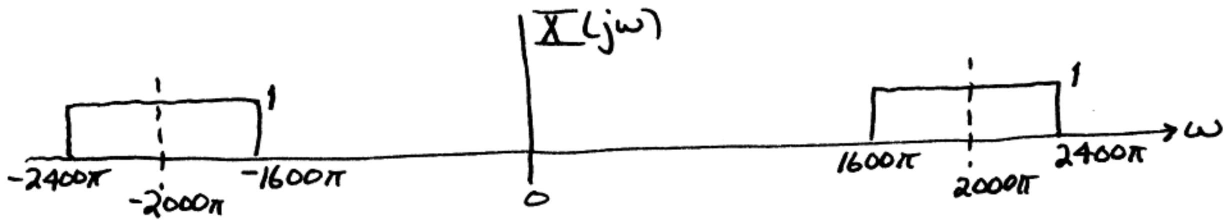


Thus
$$X(j\omega) = \begin{cases} j10\omega & \text{if } |\omega| \le 200\pi \\ 0 & \text{if } |\omega| > 200\pi \end{cases}$$

or, $X(j\omega) = j10\omega\left[u(\omega+200\pi) - u(\omega-200\pi)\right]$

(b) Multiply by cosine $\Longrightarrow$ frequency shifting

$$x(t) = 2\,\underbrace{\frac{\sin(400\pi t)}{\pi t}}_{\text{F.T. is a rectangle}} \left\{ \tfrac{1}{2}\underbrace{e^{j2000\pi t}}_{\text{shift to } \omega = 2000\pi} + \tfrac{1}{2}\underbrace{e^{-j2000\pi t}}_{\text{shift to } \omega = -2000\pi} \right\}.$$



(c) The Fourier Transform of an impulse train in time is a (different) impulse train in frequency.

$T = 10$ secs from the definition of $x(t)$.

$$\Rightarrow \quad X(j\omega) = \frac{2\pi}{10}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{10}\right)$$

spacing is $\frac{\pi}{5}$ rads.

(d) $e^{-j3\omega}$ corresponds to a time shift of 3 secs.

Inverse F.T. of $\frac{1}{2+j\omega}$ is $e^{-2t}u(t)$

$$\therefore \quad x(t) = e^{-2(t-3)}u(t-3)$$

(e) $(j\omega)$ corresponds to differentiation.

$$x(t) = \frac{d}{dt}\left\{e^{-2t}u(t)\right\} = -2e^{-2t}u(t) + \underbrace{e^{-2t}}_{\text{eval at } t=0}\delta(t) = -2e^{-2t}u(t) + \delta(t)$$

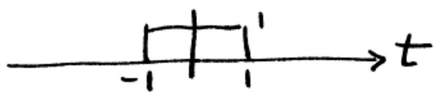
(f) Differentiation and Time-Shifting, so we can time-shift the result from part (e).

$$x(t) = -2e^{-2(t-3)}u(t-3) + \delta(t-3)$$

(g) Inverse F.T. the two parts and then convolve:

$$\frac{2\sin(\omega)}{\omega} \longrightarrow \frac{\sin(\omega T/2)}{\omega/2} \quad \text{with } T = 2 \text{ secs.}$$

$\Rightarrow$ Inverse F.T. is a pulse of length 2.



call this $p(t)$

Inverse F.T. of $\frac{2\pi}{10}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{10}\right)$

is $\sum_{n=-\infty}^{\infty}\delta(t-10n)$

$\boxed{\text{use part (d)}}$

Now, do the convolution:

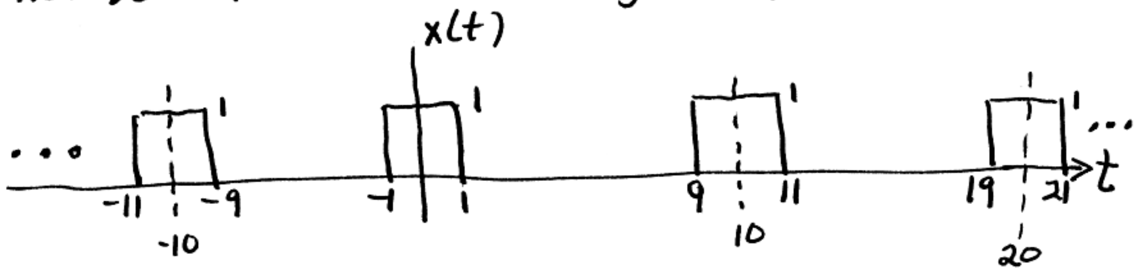
$$x(t) = p(t) * \sum_{n=-\infty}^{\infty}\delta(t-n10)$$

$$= \sum_{n=-\infty}^{\infty} p(t) * \delta(t-10n)$$

$$= \sum_{n=-\infty}^{\infty} p(t-10n)$$

Convolving with a shifted impulse will shift $p(t)$

Thus, a plot of $x(t)$ will consist of an infinite number of shifted rectangular pulses:



Prob 11.4

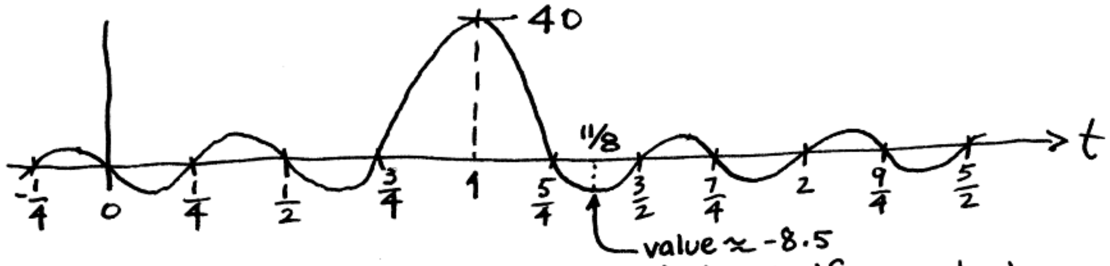
(a) $h(t)$ is a shifted "sinc": $10\dfrac{\sin 4\pi t}{\pi t}$ shifted by 1 sec.

For the "sinc", the zero crossings are at $\frac{1}{4}, \frac{1}{2}, \frac{3}{4} \ldots$

At $t=0$, the "sinc" is $10 \cdot 4\pi/\pi = 40$.

$$\lim_{t\to 0} 10\frac{\sin 4\pi t}{\pi t} = \lim_{t\to 0} 10\frac{4\pi t}{\pi t} = 40$$

Use small angle approximation $\sin(\epsilon) \approx \epsilon$

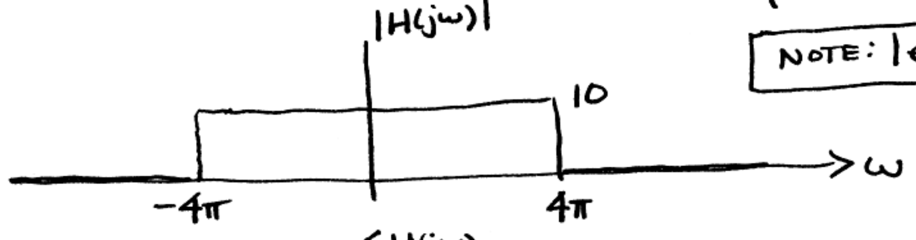


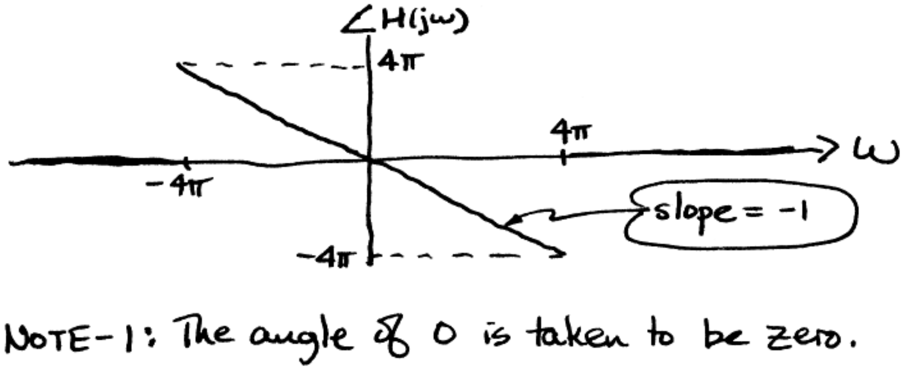
The smaller peaks are approximately halfway between the zero crossings. For example, the value at $t = \frac{11}{8}$ secs is $10\dfrac{\sin(4\pi \cdot 3/8)}{\pi(3/8)} = 10\dfrac{\sin(3\pi/2)}{3\pi/8} = -\dfrac{80}{3\pi} \approx -8.5$

(b) The F.T. of a "sinc" is a rectangular pulse.
Time-shifting by 1 sec. corresponds to $e^{-j\omega}$

$$\Rightarrow H(j\omega) = 10e^{-j\omega}\{u(\omega+4\pi) - u(\omega-4\pi)\} = \begin{cases} 10e^{-j\omega} & |\omega| \le 4\pi \\ 0 & |\omega| > 4\pi \end{cases}$$

$\boxed{\text{NOTE: } |e^{-j\omega}| = 1}$





NOTE-1: The angle of 0 is taken to be zero.

NOTE-2: If the phase were plotted in MATLAB or evaluated with an ArcTangent function, it would exhibit jumps of 2π because ArcTan always gives an answer between $-\pi$ and $+\pi$.

Prob 11.5

(a) Since the system is LINEAR, the two inputs can be treated separately and then combined.

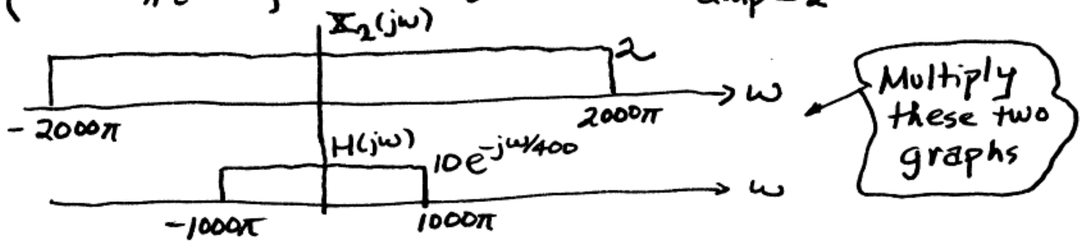
- For the cosine input, the output will be a cosine with a new magnitude and phase. We evaluate $H(j\omega)$ at the input frequency: $\omega = 200\pi$ rad/s.

$$H(j200\pi) = 10e^{-j(200\pi)(0.0025)} = 10e^{-j0.5\pi}$$

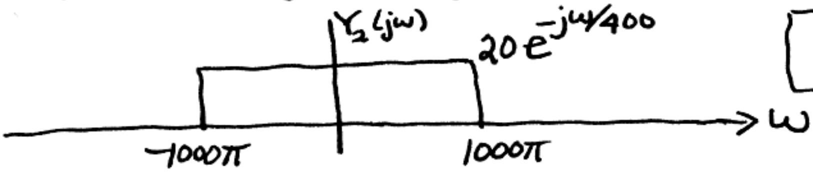
Call this output $y_1(t)$: $\quad y_1(t) = 10\cos(200\pi t - \pi/2)$

- For the "sinc" input, take the F.T. of the input, then multiply by $H(j\omega)$ and then inverse transform.

$$\text{F.T.}\left\{2\,\frac{\sin(2000\pi t)}{\pi t}\right\} = \text{Rectangular Shape: width of } 4000\pi \text{ rad/s. amp} = 2$$



Thus $Y_2(j\omega) = H(j\omega)X_2(j\omega)$ is also a rectangle



$\boxed{\frac{1}{400} = 0.0025\text{s}}$

The inverse F.T. of this rectangle is a <u>SHIFTED</u> "sinc"

$$y_2(t) = 20\,\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

Finally, the total output is the sum of $y_1(t)$ & $y_2(t)$

$$y(t) = 10\cos(200\pi t - \pi/2) + 20\,\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

We have used <u>SUPERPOSITION</u> to do this part.

Prob 11.5

(b) Use SUPERPOSITION again. Two of the inputs are the same, so we don't have to rework them. We only need to consider the input $x_3(t) = \cos(3000\pi t)$.
For a cosine input, we must evaluate $H(j\omega)$ at the input frequency; in this case, at $\omega = 3000\pi$.

$H(j3000\pi) = 0 \quad \Rightarrow$ NO OUTPUT, i.e. $y_3(t) = 0$.

So, the answer is the same as part (a)!

(c) Again, use SUPERPOSITION. We already know the output for $x_1(t) = \cos(200\pi t)$

$$y_1(t) = 10\cos(200\pi t - \pi/2)$$

We need to find the output for $x_4(t) = 2\delta(t)$.
Thus we need the impulse response
But this is just the inverse F.T. of $H(j\omega)$
And we already know that is a shifted "sinc"

$$y_4(t) = 2h(t) = 20\,\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

Finally,

$$y(t) = y_1(t) + y_4(t)$$
$$= 10\cos(200\pi t - \pi/2) + 20\,\frac{\sin(1000\pi(t - 1/400))}{\pi(t - 1/400)}$$

It is very interesting to see that all three parts have the same answer. Why? Because the filter is an ideal LPF, so only the part of the input signal between -1000π and $+1000\pi$ matters. For example, in part (c) the F.T. of $2\delta(t)$ is $X_4(j\omega) = 2$ for all ω, but only the part for $|\omega| < 1000\pi$ rad/s matters. Over that range the "sinc" input of part (a) is the same

(d) Superposition simplifies the work.

Prob 11.6

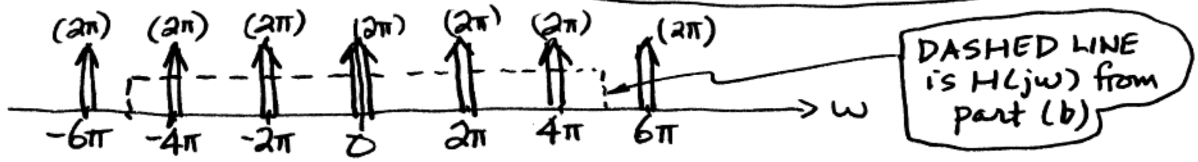
(a) The F.T. of an impulse train is another impulse train.

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n) = \sum_{k=-\infty}^{\infty} e^{j2\pi k t}$$

Each complex exp transforms to $2\pi\delta(\omega - k\omega_0)$

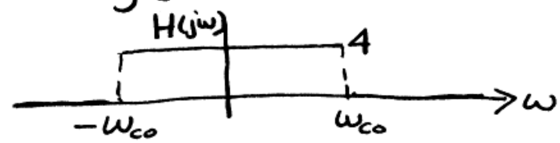
$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

NOTE: $\omega_0 = 2\pi$ rad/s



(b) The F.T. of a "sinc" is a rectangle

$$h(t) = 4\,\frac{\sin(\omega_{co} t)}{\pi t} \longrightarrow$$



(c) $Y(j\omega)$ will consist of 5 impulses because $H(j\omega)=0$ when $|\omega| > 5\pi$

$$Y(j\omega) = 8\pi\delta(\omega) + 8\pi\delta(\omega-2\pi) + 8\pi\delta(\omega+2\pi) + 8\pi\delta(\omega-4\pi) + 8\pi\delta(\omega+4\pi)$$

Each $\delta(\omega - ?)$ will inverse F.T. to an exponential.

$$y(t) = 4 + 4e^{j2\pi t} + 4e^{-j2\pi t} + 4e^{j4\pi t} + 4e^{-j4\pi t}$$

$$y(t) = 4 + 8\cos(2\pi t) + 8\cos(4\pi t) \quad \text{for \underline{all} } t$$

(d) If you want only the constant term then put the cutoff frequency of the filter below 2π rad/s, but keep it greater than ZERO.

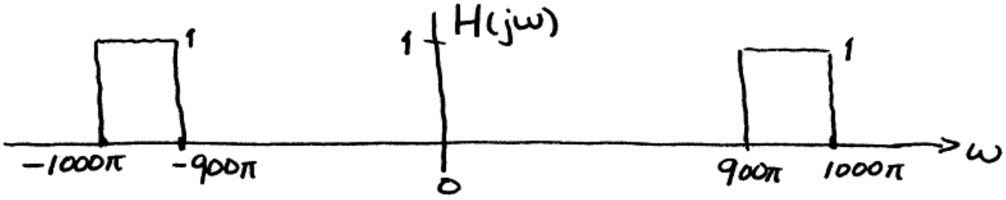
$$\boxed{0 < \omega_{co} < 2\pi}$$

In this case, $Y(j\omega) = 8\pi\delta(\omega)$

$$\Rightarrow y(t) = 4 \qquad \boxed{c = 4}$$

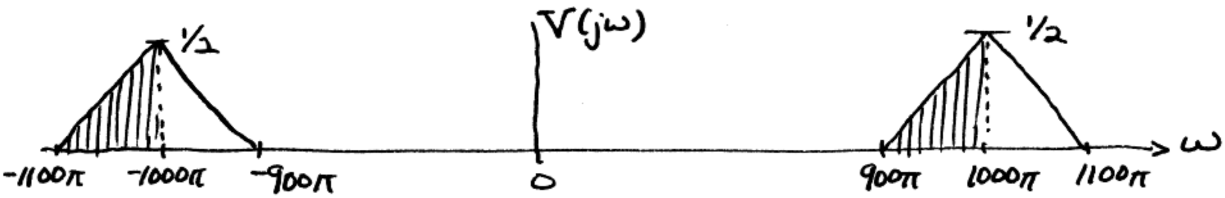
Prob 11.7

(a)

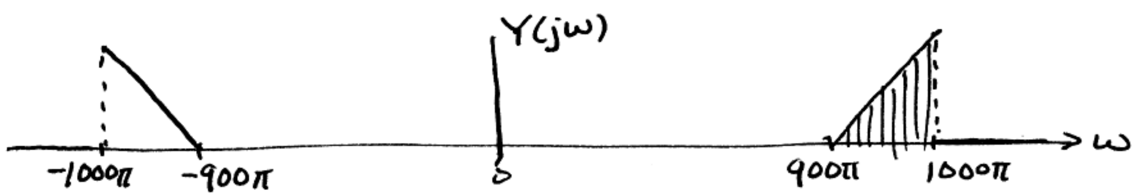


(b) Multiplying by a cosine will cause "frequency shifting"

$$V(j\omega) = \frac{1}{2}X(j(\omega+1000\pi)) + \frac{1}{2}X(j(\omega-1000\pi))$$



(c) Multiply the two graphs



(d) The term "single sideband" (SSB) comes from the following observations:

1. The original $X(j\omega)$ has a positive frequency portion and a negative frequency portion (shaded)
2. In $V(j\omega)$ the shaded portion is called the lower sideband because it is below the "carrier frequency" of 1000π rad/s in this case.
3. The final spectrum (i.e., F.T.) for $Y(j\omega)$ has only one of the sidebands for $w>0$ and the other one for $w<0$. Thus a single sideband remains