Prob 10.1

(a) $$H(j\omega) = \int_{-\infty}^{\infty} \{\delta(t) - 0.1e^{-0.1t}u(t)\} e^{-j\omega t}\,dt$$

$$= \underbrace{\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\,dt}_{= e^{-j\omega(0)} = 1} - 0.1 \underbrace{\int_{-\infty}^{\infty} e^{-0.1t}u(t)e^{-j\omega t}\,dt}$$

$$\int_0^{\infty} e^{-0.1t}e^{-j\omega t}\,dt = \left.\frac{e^{-(0.1+j\omega)t}}{-(0.1+j\omega)}\right|_0^{\infty} = 0 - \frac{1}{-(0.1+j\omega)} = \frac{1}{0.1+j\omega}$$

Thus, $$H(j\omega) = 1 - \frac{0.1}{0.1+j\omega} = \frac{j\omega}{0.1+j\omega}$$

(b) $$|H(j\omega)|^2 = \left(\frac{j\omega}{0.1+j\omega}\right)\left(\frac{-j\omega}{0.1-j\omega}\right) = \frac{\omega^2}{0.01 + j0.1\omega - j0.1\omega - (j\omega)^2}$$

$$= \frac{\omega^2}{0.01+\omega^2}$$

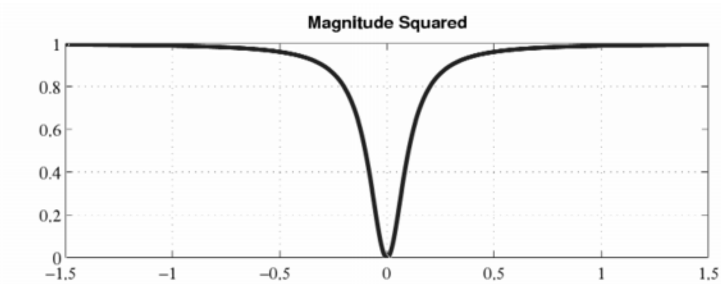
At $\omega = 0$, $|H(j0)|^2 = 0$

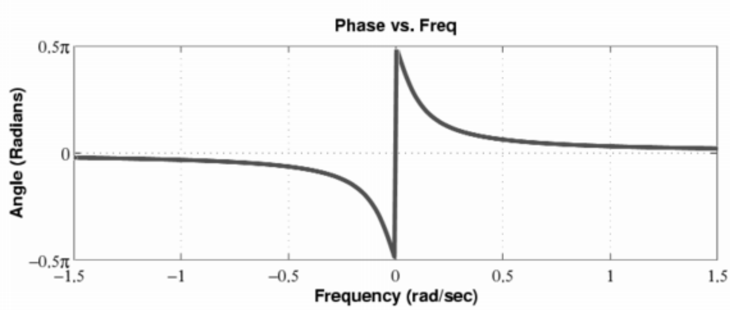
At $\omega = \infty$, $|H(j\infty)|^2 = \lim_{\omega\to\infty} \frac{\omega^2}{0.01+\omega^2} = \lim_{\omega\to\infty} \frac{\omega^2}{\omega^2} = 1$

At $\omega = 0.1$, $|H(j0.1)|^2 = \frac{0.01}{0.01+0.01} = \frac{1}{2}$

$$\angle H(j\omega) = \angle j\omega - \angle(0.1+j\omega) = \begin{cases} \pi/2 - \text{ArcTan}\left(\frac{\omega}{0.1}\right) & \text{if } \omega > 0 \\ -\pi/2 - \text{ArcTan}\left(\frac{\omega}{0.1}\right) & \text{if } \omega < 0 \end{cases}$$

Plots from MATLAB are below:





Prob 10.1 (cont)

(c) From the plot in part (b), the max value is one as $\omega \to \infty$. Also $|H(j\omega)|^2 = 1/2$ at $\omega = 0.1$ rad/s.

Why is it called "3dB point"?

$$10\log_{10}|H(j0.1)|^2 = 10\log_{10}\left(\tfrac{1}{2}\right) = 10(-0.301) = -3.01\,dB$$

Notice that $10\log_{10}|H(j\infty)|^2 = 10\log_{10}(1) = 0$, so the decibel value at $\omega = 0.1$ rad/s is -3.01 dB down from the maximum dB value.

(d) Use <u>SUPERPOSITION</u> to do each input separately and then add them together.

$$x(t) = \underbrace{10}_{x_1(t)} + \underbrace{20\cos(0.1t)}_{x_2(t)} + \underbrace{\delta(t-0.2)}_{x_3(t)}$$

① $x_1(t)$ is a sinusoid whose frequency is zero. Thus we need $H(j\omega)$ at $\omega=0$. $H(j0) = \dfrac{j0}{0.1+j0} = 0$

$\Rightarrow y_1(t) = 0$

② $x_2(t)$ is a sinusoid with $\omega = 0.1$ rad/s.

$H(j\omega)$ at $\omega = 0.1$ is $H(j0.1) = \dfrac{j0.1}{0.1+j0.1} = \dfrac{j}{1+j}$

We need $H(j0.1)$ in <u>POLAR</u> form.

$$H(j0.1) = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{j+1}{2} = \frac{\sqrt{2}}{2}e^{j\pi/4}$$

$$\Rightarrow y_2(t) = \left(\frac{\sqrt{2}}{2}\right)20\cos(0.1t+\pi/4) = 10\sqrt{2}\cos(0.1t+\pi/4)$$

③ for $x_3(t)$ we have a shifted impulse, so use $h(t)$.

$$y_3(t) = h(t-0.2) = \delta(t-0.2) - 0.1e^{-0.1(t-0.2)}u(t-0.2)$$

Now, add them together:

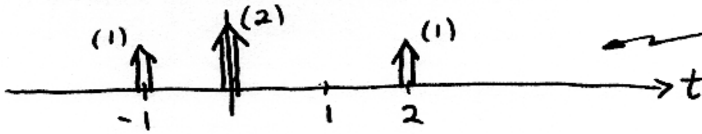
$$y(t) = y_1(t) + y_2(t) + y_3(t)$$

$$y(t) = 10\sqrt{2}\cos(0.1t+\pi/4) + \delta(t-0.2) - 0.1e^{-0.1(t-0.2)}u(t-0.2)$$

Prob 10.2

(a) Let $x(t) = \delta(t)$

Then $y(t) = \delta(t+1) + 2\delta(t) + \delta(t-2)$ ← This is $h(t)$



← Plot of $h(t)$

(b) $$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \{\delta(t+1) + 2\delta(t) + \delta(t-2)\} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t+1) e^{-j\omega t} dt + \int_{-\infty}^{\infty} 2\delta(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} \delta(t-2) e^{-j\omega t} dt$$

$$H(j\omega) = e^{-j\omega(-1)} + 2e^{-j\omega(0)} + e^{-j\omega(2)}$$

$$= e^{j\omega} + 2 + e^{-j2\omega}$$

(c) Let $x(t) = e^{j\omega t}$

Then $$y(t) = e^{j\omega(t+1)} + 2e^{j\omega t} + e^{j\omega(t-2)}$$

$$= e^{j\omega t} e^{j\omega} + 2e^{j\omega t} + e^{j\omega t} e^{-j2\omega}$$

$$= \underbrace{\{e^{j\omega} + 2 + e^{-j2\omega}}\} e^{j\omega t}$$

This is $H(j\omega)$ which is the <u>same</u> as in part (b).

Prob 10.3

(a) The Fourier Transform (FT) of $\delta(t)$ is 1. Thus, the FT of $\delta(t-t_d)$ is $1e^{-j\omega t_d}$

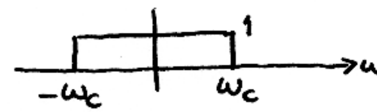
using linearity of the FT.

$$FT\{x(t)\} = FT\{\delta(t+1)\} + FT\{2\delta(t)\} + FT\{\delta(t-1)\}$$

$$X(j\omega) = e^{j\omega} + 2 + e^{-j\omega}$$

$$= 2 + 2\cos\omega$$ if you simplify

(b) $\dfrac{\sin(100\pi(t-2))}{\pi(t-2)}$ is a shifted "sinc" function

The FT of $\dfrac{\sin(\omega_c t)}{\pi t}$ is a rectangle 

This F.T. can be found in Table 12.1.

It can also be written in terms of unit steps as $u(\omega+\omega_c) - u(\omega-\omega_c)$. In this case, $\omega_c = 100\pi$ rad/s

Using the shift property with $t_d = 2$

$$X(j\omega) = \{u(\omega+100\pi) - u(\omega-100\pi)\}e^{-j2\omega}$$

(c) The F.T. of $e^{-at}u(t)$ is $\dfrac{1}{a+j\omega}$.

$$x(t) = e^{-t}u(t) - \underbrace{e^{-t}u(t-4)}_{\text{This is NOT a pure shift}} = e^{-t}u(t) - e^{-4}\underbrace{e^{-(t-4)}u(t-4)}_{\text{This re-write shows the shift of both terms}}$$

$$X(j\omega) = \frac{1}{1+j\omega} - e^{-4}\frac{e^{-j4\omega}}{1+j\omega}$$

$$= \frac{1 - e^{-4(1+j\omega)}}{1+j\omega}$$

Prob 10.4

The general approach is to use Tables 12.1 & 12.2 plus some algebraic manipulations:

(a) $\frac{j\omega}{0.1+j\omega} e^{-j0.2\omega} = X_1(j\omega) e^{-j0.2\omega}$ ← use time shifting

If $X_1(j\omega) = \frac{j\omega}{0.1+j\omega}$, then $X_1(j\omega) = j\omega X_2(j\omega)$ ← use derivative

If $X_2(j\omega) = \frac{1}{0.1+j\omega} \Rightarrow x_2(t) = e^{-0.1t}u(t)$

$\Rightarrow x_1(t) = \frac{d}{dt} x_2(t) = e^{-0.1t}\delta(t) - 0.1e^{-0.1t}u(t) = \delta(t) - 0.1e^{-0.1t}u(t)$

$x(t) = x_1(t-0.2) = \delta(t-0.2) - 0.1e^{-0.1(t-0.2)}u(t-0.2)$

(b) $X(j\omega) = 2 + 2\cos\omega = 2 + e^{-j\omega} + e^{j\omega}$ ← use shifting

$x(t) = 2\delta(t) + \delta(t-1) + \delta(t+1)$

(c) use Table entry $\frac{1}{a+j\omega} \longrightarrow e^{-at}u(t)$

$x(t) = e^{-t}u(t) - e^{-2t}u(t)$

(d) use Table entry: $2\pi\delta(\omega-\omega_0) \longrightarrow e^{j\omega_0 t}$

$X(j\omega) = j\frac{2\pi}{2\pi}\delta(\omega - 100\pi) - j\frac{2\pi}{2\pi}\delta(\omega-(-100\pi))$

$x(t) = \frac{j}{2\pi}e^{j100\pi t} - \frac{j}{2\pi}e^{-j100\pi t}$

$= \frac{-1}{\pi}\left\{\frac{1}{2j}e^{j100\pi t} - \frac{1}{2j}e^{-j100\pi t}\right\}$ ← use Inverse Euler

$x(t) = -\frac{1}{\pi}\sin(100\pi t)$

Prob 10.5

(a) The period is $T_0 = 8$, so $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8} = \frac{\pi}{4}$ rad/s

$$a_k = \frac{1}{8}\int_{-1}^{1} 10 e^{-j\frac{\pi}{4}kt}\,dt$$

The limits on the integral are NOT -4 to $+4$ because $x(t)$ is ZERO for $-4 \le t < -1$ and $1 < t \le 4$.

(b) To plot the spectrum, we need the values of a_k for $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

At $k=0$ use L'Hôpital's rule or take $\lim_{k\to 0}$

$$a_0 = \frac{10\,(\pi k/4)}{\pi k} = \frac{10}{4} = 2.5$$

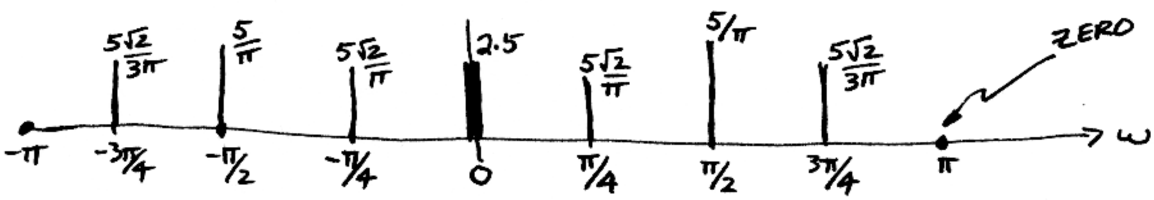
$$a_1 = \frac{10\sin(\pi/4)}{\pi} = \frac{10\cdot\sqrt{2}/2}{\pi} = \frac{5\sqrt{2}}{\pi}$$

NOTE: $a_{-1} = a_1$ and generally we have $a_k = a_{-k}$

$$a_2 = \frac{10\sin(\pi/2)}{2\pi} = \frac{10}{2\pi} = \frac{5}{\pi} = a_{-2}$$

$$a_3 = \frac{10\sin(3\pi/4)}{3\pi} = \frac{10\sqrt{2}/2}{3\pi} = \frac{5\sqrt{2}}{3\pi} = a_{-3}$$

$$a_4 = \frac{10\sin(\pi)}{\pi k} = 0$$

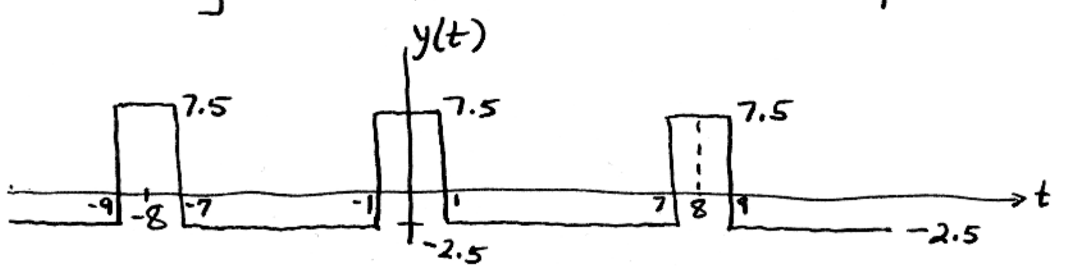


(c) The frequency response of the filter will MULTIPLY the spectrum of the input. Thus the spectrum of the output will be everything EXCEPT the line at DC.

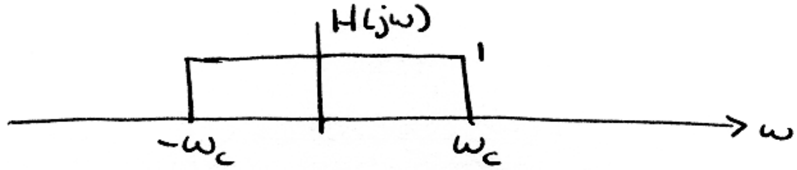
Thus $y(t)$ has a FOURIER Series that is identical to the FS for $x(t)$ except the a_0 term is missing

$$\Rightarrow \quad y(t) = x(t) - a_0 = x(t) - 2.5$$

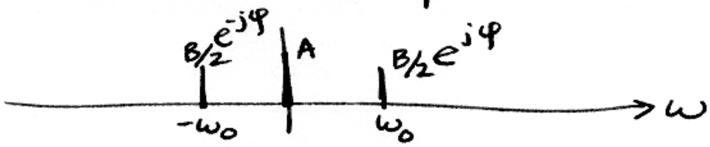
Subtracting a constant will shift the plot down



(d)

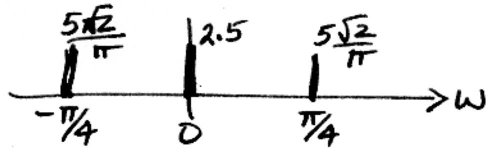


Again, we note that $H(j\omega)$ will <u>MULTIPLY</u> the spectrum of $x(t)$. We want the spectrum of the output to be



Since $\omega_0 = \pi/4$, we need $\omega_c > \omega_0$. But we also need $\omega_c < 2\omega_0$. Thus $\frac{\pi}{4} < \omega_c < \frac{\pi}{2}$

With this ω_c, the spectrum of $y(t)$ will be:



$\Rightarrow$

$$y(t) = 2.5 + \frac{10\sqrt{2}}{\pi}\cos\left(\frac{\pi}{4}t\right)$$

(e) If $H(j\omega) = 1 - e^{-j2\omega}$ we can find $h(t)$ by doing an INVERSE FOURIER TRANSFORM.

$$h(t) = \delta(t) - \delta(t-2)$$

Then $y(t) = x(t) * h(t) = x(t) * \delta(t) - x(t) * \delta(t-2)$

$= x(t) - x(t-2)$.

So we must shift $x(t)$ by 2 and then subtract