1-Feb-01 ECE 2025 Spring-2001

Prob 4.1

(a) Use Euler's inverse formula

$$x(t) = \left[10 + \tfrac{5}{2}e^{j(2000\pi t + \pi/5)} + \tfrac{5}{2}e^{-j(2000\pi t + \pi/5)}\right]\left[\tfrac{1}{2}e^{j10000\pi t} + \tfrac{1}{2}e^{-j10000\pi t}\right]$$

$$= 5e^{j10000\pi t} + \tfrac{5}{4}e^{j\pi/5}e^{j12000\pi t} + \tfrac{5}{4}e^{-j\pi/5}e^{j8000\pi t}$$

$$+ 5e^{-j10000\pi t} + \tfrac{5}{4}e^{j\pi/5}e^{-j8000\pi t} + \tfrac{5}{4}e^{-j\pi/5}e^{-j12000\pi t}$$

(see part (d) for the Fourier Series form)

(b) The complex exponentials have frequencies:

$$\{10{,}000\pi,\ 12{,}000\pi,\ 8000\pi,\ -10000\pi,\ -8000\pi,\ -12000\pi\}$$

The greatest common divisor is 2000π, so $\boxed{\omega_0 = 2000\pi \text{ rad/s}}$

(c) There is no component at $\omega = 0$, so $\boxed{\text{DC-value} = 0}$

(d) We need to know which harmonics are present: $\frac{10000\pi}{\omega_0} = 5$, $\frac{12000\pi}{\omega_0} = 6$, $\frac{8000\pi}{\omega_0} = 4$

So we have $a_4, a_5, a_6, a_{-4}, a_{-5}, a_{-6}$

$a_4 = \frac{5}{4}e^{-j\pi/5}$ $\quad a_{-4} = \frac{5}{4}e^{j\pi/5}$ $\quad$ (Note: $a_4^* = a_{-4}$)

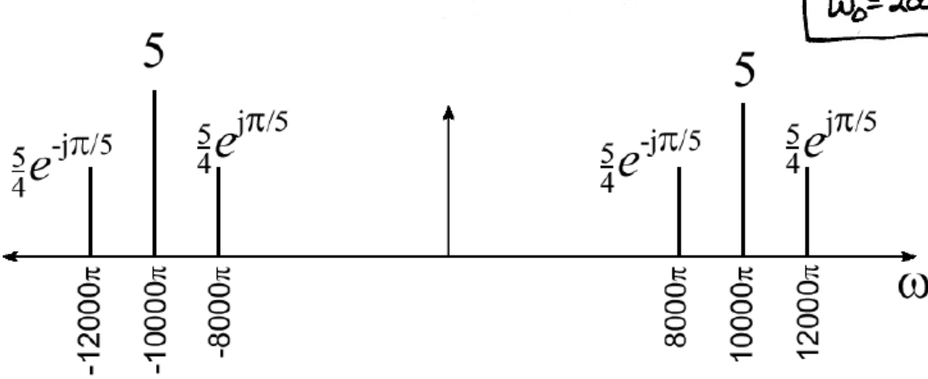
$a_5 = 5$ $\quad a_{-5} = 5$

$a_6 = \frac{5}{4}e^{j\pi/5}$ $\quad a_{-6} = \frac{5}{4}e^{-j\pi/5}$ $\quad$ (Note: $a_5^* = a_{-5}$)

The Fourier Series is:

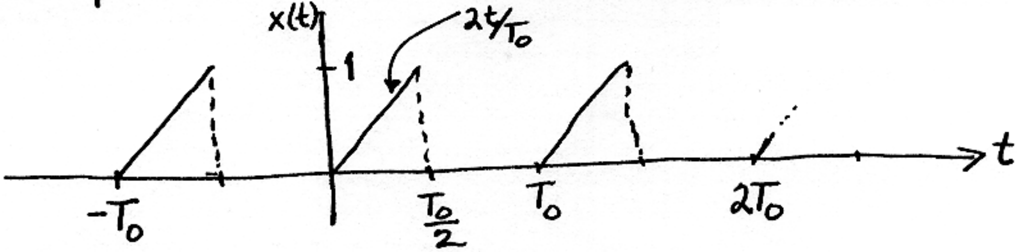
$$x(t) = a_4 e^{j4\omega_0 t} + a_5 e^{j5\omega_0 t} + a_6 e^{j6\omega_0 t} + a_{-4}e^{-j4\omega_0 t} + a_{-5}e^{-j5\omega_0 t} + a_{-6}e^{-j6\omega_0 t}$$

$\boxed{\omega_0 = 2000\pi \text{ rad/s}}$



Prob 4.2

(a) The period of $x(t)$ is T_0, so $x(t+T_0) = x(t)$.



(b) $$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0/2} \left(2t/T_0\right) dt = \frac{1}{T_0^2}\,\frac{2t^2}{2}\Bigg|_0^{T_0/2}$$

$$a_0 = \frac{1}{T_0^2}\left(T_0^2/4\right) = 1/4$$

Alternate calculation: $\frac{1}{T_0}\{\text{Area under one period}\}$

$$a_0 = \frac{1}{T_0}\left\{1\left(\frac{T_0}{2}\right)\cdot\frac{1}{2}\right\} = 1/4$$

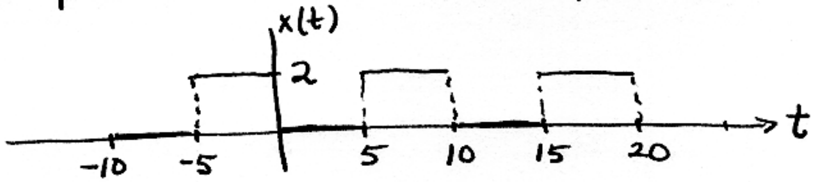
(c) Take advantage of the fact that $x(t)=0$ for half of the period from $\frac{T_0}{2}$ to T_0.

$$a_k = \frac{1}{T_0}\int_0^{T_0/2}\left(\frac{2t}{T_0}\right) e^{-jk\left(\frac{2\pi}{T_0}\right)t}\,dt$$

$\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency.

Prob 4.3

(a) The period is 10 s, so the plot covers 3 periods



(b) $a_0 = \frac{1}{T_0}\{\text{Area}\} = \frac{1}{10}\{2 \cdot 5\} = 1$

Another interpretation, a_0 is the average value.

(c) $$a_k = \frac{1}{10}\int_5^{10} 2e^{-jk\omega_0 t}\,dt \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{10} = \frac{\pi}{5} \text{ rad/s}$$

$$a_1 = \frac{1}{10}\int_5^{10} 2e^{-j\omega_0 t}\,dt \qquad k=1$$

$$= \frac{1}{5}\left.\frac{e^{-j\omega_0 t}}{-j\omega_0}\right|_5^{10} = \frac{1}{-j5\omega_0}\left\{e^{-j10\omega_0} - e^{-j5\omega_0}\right\}$$

$$a_1 = \frac{1}{-j\pi}\left\{e^{-j2\pi} - e^{-j\pi}\right\} = \frac{1}{-j\pi}\{1-(-1)\} = \frac{2}{-j\pi} = \frac{2}{\pi}e^{j\pi/2}$$

(d) $$y(t) = 1 + x(t) = 1 + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= 1 + a_0 + \sum_{k\neq 0} a_k e^{jk\omega_0 t}$$

$$\Rightarrow b_0 = 1 + a_0 = 2$$

$$b_1 = a_1$$

Prob 4.4

(a) $y(t) = Ax(t) = A\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\}$

$$= \sum_{k=-\infty}^{\infty} (Aa_k) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$\Rightarrow b_k = Aa_k$$

(b) $y(t) = x(t-t_d) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_d)}$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_d}$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_d}\right) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$b_k = a_k e^{-jk\omega_0 t_d}$$

Prob 4.5

(a) $a_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} 1\, dt = \frac{1}{T_0}\left\{\frac{T_0}{4} - \left(-\frac{T_0}{4}\right)\right\} = \frac{1}{2}$

$$a_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} 1 e^{-jk\omega_0 t}\, dt \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$= \frac{1}{T_0}\left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right|_{-T_0/4}^{T_0/4} = \frac{1}{-jk\omega_0 T_0}\left\{e^{-jk\omega_0 \frac{T_0}{4}} - e^{-jk\omega_0(-T_0/4)}\right\}$$

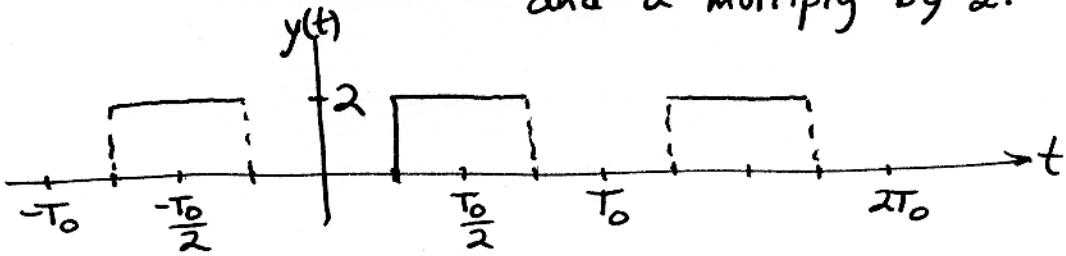
Since $\omega_0 T_0 = 2\pi$, we can simplify:

$$a_k = \frac{1}{-j2\pi k}\left\{e^{-jk\pi/2} - e^{jk\pi/2}\right\} = \frac{1}{\pi k}\left\{\frac{e^{jk\pi/2} - e^{-jk\pi/2}}{2j}\right\}$$

use inverse Euler

$$= \frac{\sin(k\pi/2)}{\pi k} \quad \text{for } k \neq 0$$

(b) $y(t) = 2x(t - T_0/2)$ is a shift to the right by $T_0/2$ and a multiply by 2.



Apply problem 4.4 with $A = 2$ and $t_d = T_0/2$.

$$b_k = a_k \cdot 2e^{-jk\omega_0 T_0/2} = a_k \cdot 2e^{-jk\pi}$$

$$b_0 = 2a_0 = 2\left(\tfrac{1}{2}\right) = 1$$

$$b_k = \frac{\sin(k\pi/2)}{\pi k} \cdot 2e^{-jk\pi}$$

Note:
$e^{-jk\pi} = (e^{-j\pi})^k$
$= (-1)^k$