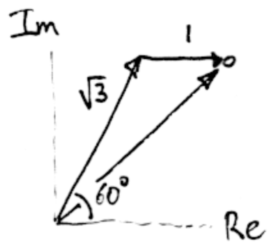
Solutions P.S. #2

2.2. $x(t) = \sqrt{3}\cos\left(\omega_0 t + \frac{\pi}{3}\right) + \sin\left(\omega_0 t + \frac{\pi}{2}\right)$

a) $x(t) = \mathrm{Re}\left[\sqrt{3}\, e^{j[\omega_0 t + \frac{\pi}{3}]} + e^{j[(\omega_0 t + \frac{\pi}{2}) - \frac{\pi}{2}]}\right]$

<u>Note</u>: $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$
was used here!



$$= \mathrm{Re}\left[\left(\sqrt{3}\, e^{j\pi/3} + 1\right) e^{j\omega_0 t}\right]$$

$$= \mathrm{Re}\left[\left(\sqrt{3}\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) + 1\right) e^{j\omega_0 t}\right]$$

$$= \mathrm{Re}\left[\left(\sqrt{3}\left(\frac{1}{2}\right) + j\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) + 1\right) e^{j\omega_0 t}\right]$$

$$= \mathrm{Re}\left[\sqrt{\left(\frac{\sqrt{3}}{2}+1\right)^2 + \left(\frac{3}{2}\right)^2}\; e^{j\theta} \cdot e^{j\omega_0 t}\right]$$

where $\tan\theta = \dfrac{3/2}{\sqrt{3}/2 + 1}$, $\theta \in$ first quadrant since $\begin{cases} \sin\theta > 0 \\ \cos\theta > 0 \end{cases}$

$$\therefore\; x(t) = \underbrace{\frac{\sqrt{(\sqrt{3}+2)^2 + 3^2}}{2}}_{A} \cos(\omega_0 t + \theta)$$

$$A = \frac{\sqrt{3 + 4\sqrt{3} + 4 + 9}}{2} = \frac{\sqrt{16 + 4\sqrt{3}}}{2} = \sqrt{4 + \sqrt{3}} = 2.394$$

$$\tan\theta = \frac{3}{\sqrt{3}+2} \rightarrow \theta = 39.79^\circ \; (.677 \text{ rad})$$

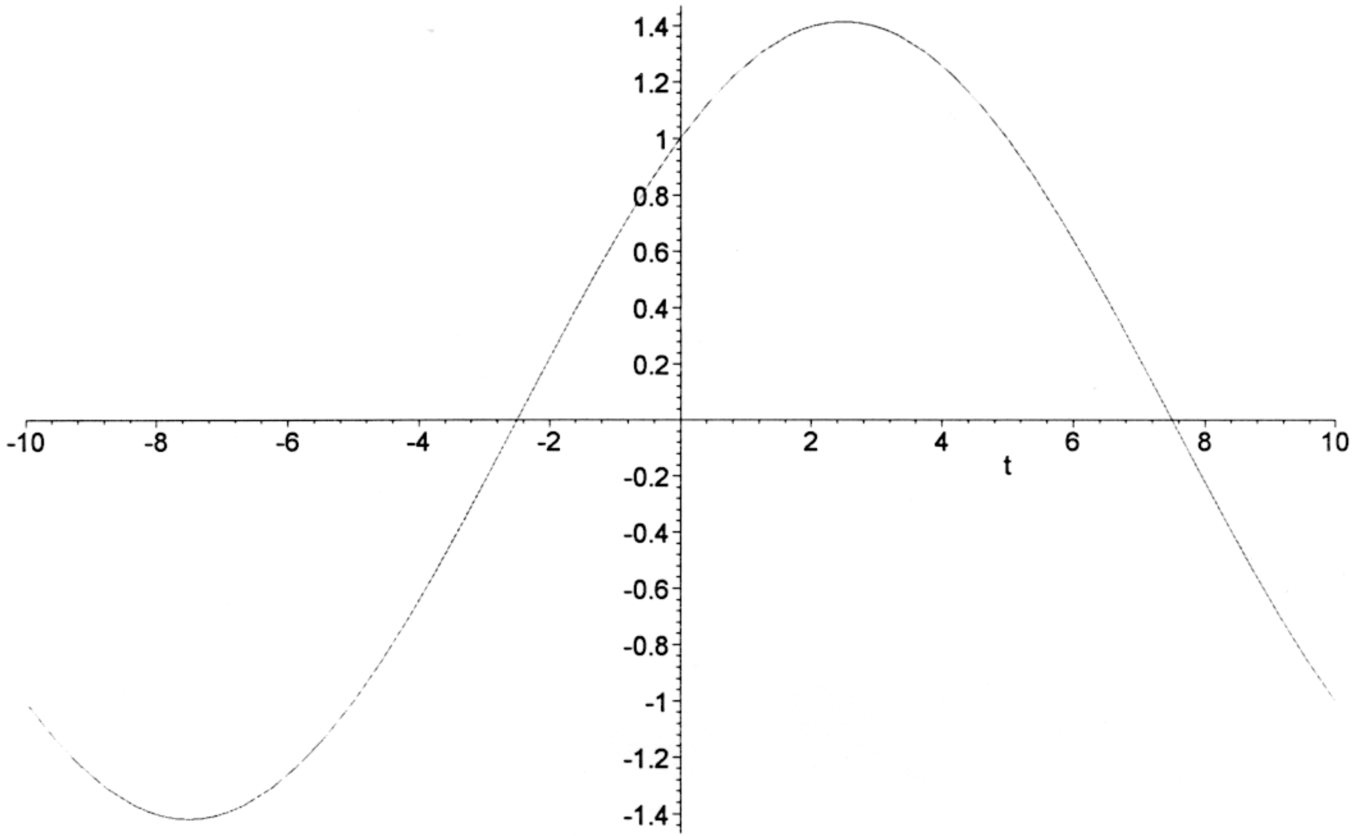
$$\Rightarrow\; x(t) = \sqrt{4+\sqrt{3}}\cos(\omega_0 t + .677)$$

$$z(t) = 2.394 e^{j(\omega_0 t + 0.677)}$$

b) $\mathrm{Re}\left((1-j)e^{j\omega_0 t}\right) = \mathrm{Re}\ \sqrt{2}\, e^{j(\omega_0 t - \pi/4)} = \sqrt{2}\cos(\omega_0 t - \pi/4)$

$\Delta t = 20$ $\qquad = \sqrt{2}\cos\left(\frac{\pi t}{10} - \pi/4\right)$

$\omega_0 \cdot \Delta t = (0.1)\pi \cdot 20 = 2\pi \rightarrow 1$ period



max at $t_0 = 2.5 = \frac{10}{4} \Rightarrow \sqrt{2}\cos\left(\frac{\pi t}{10} - \frac{\pi}{4}\right)$

$$= \sqrt{2}\cos\left(\frac{\pi}{10}\left(t - \frac{10}{4}\right)\right)$$

$\uparrow$ "delay time"

2.3 $$x(t) = 5\sqrt{2}\cos\left(20\pi t + \frac{\pi}{4}\right) + A\cos\left(2\pi t + \phi\right)$$
$$= B\cos(20\pi t)$$

$A > 0$

$B > 0$

Let $\underline{X}$ be the phasor representing $x(t)$

$$\underline{X} = 5\sqrt{2}\, e^{j\pi/4} + A e^{j\phi} = B$$
$$= \underbrace{5\sqrt{2}\cos\frac{\pi}{4}}_{=} + j5\sqrt{2}\sin\frac{\pi}{4} + A\cos\phi + jA\sin\phi = B$$
$$= \left(5\sqrt{2}\,\frac{\sqrt{2}}{2} + A\cos\phi\right) + j\left(5\sqrt{2}\,\frac{\sqrt{2}}{2} + A\sin\phi\right) = B$$

a) Thus: for zero phase: $\begin{cases} \text{Imaginary part zero} \\ \text{Real part positive} \end{cases}$

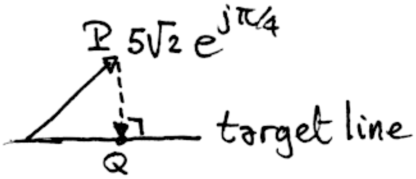
$$\boxed{\begin{array}{l} 5 + A\sin\phi = 0 \\ 5 + A\cos\phi > 0 \end{array}}$$

b) For $B = 10$, solve: $\begin{cases} 5 + A\sin\phi = 0 \\ 5 + A\cos\phi = 10 \end{cases} \Rightarrow \begin{array}{l} A\sin\phi = -5 \\ A\cos\phi = 5 \end{array}$

$$\Rightarrow \underbrace{A^2\sin^2\phi + A^2\cos^2\phi}_{=A^2} = (-5)^2 + (5)^2 = 25 + 25 = 50$$
$$\Rightarrow \boxed{A = \sqrt{50}}$$

and $\left.\begin{array}{l} \cos\phi = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} \\ \sin\phi = -\frac{5}{\sqrt{50}} = -\frac{1}{\sqrt{2}} \end{array}\right\} \rightarrow \boxed{\phi = -\pi/4}$

c)



Orthogonal projection gives PQ of minimal length. $\Rightarrow A\cos(2\pi t + \phi)$

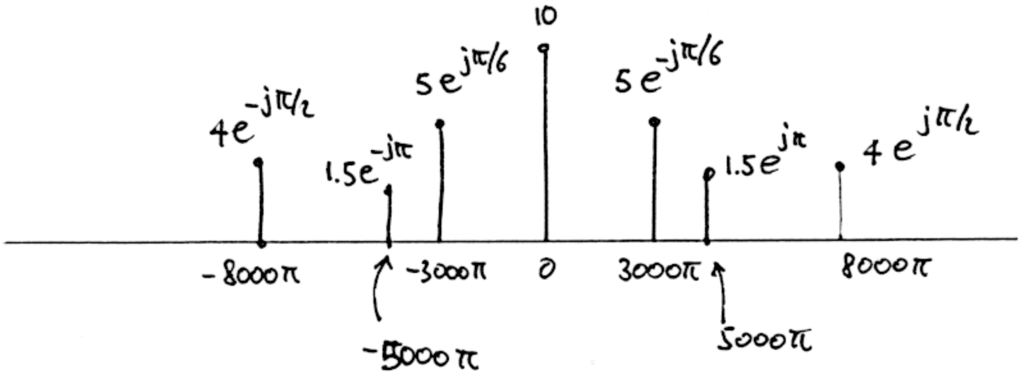
$A = 5\sqrt{2}\sin\frac{\pi}{4} = 5$, $\phi = -\pi/2 \Rightarrow B = 5$

2.4 : a) $x(t) = 5 + (2.5 \times 2)\cos\left(3000\pi t - \frac{\pi}{6}\right) + (2 \times 2)\cos\left(8000\pi t + \frac{\pi}{2}\right)$

careful! do not forget this factor 2 !

b)
$$y(t) = 2x(t) - 3\cos(5000\pi(t - 0.002))$$
$$= 2x(t) - 3\cos(5000\pi t - 10\pi)$$
$$= 2x(t) - 3\cos(5000\pi t)$$
$$= 2x(t) + 3\cos(5000\pi t + \pi)$$

$$\text{Spec}(y) = 2 \cdot \text{Spec}(x) + \text{Spec}(3\cos(5000\pi t + \pi))$$



2.5 : $y(t) = \left[3\cos\left(2000\pi t + \frac{\pi}{4}\right) + \cos\left(4000\pi t + \frac{\pi}{2}\right) + A\right]$

$$\cos 2\pi \cdot 750 \cdot 10^3 t$$

Use the identity:

$$\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

$$\Rightarrow y(t) = A\cos 2\pi \cdot 750 \cdot 10^3 t$$

$$+ \frac{3}{2}\cos\left(2\pi \cdot 750 \cdot 10^3 t + 2000\pi t + \frac{\pi}{4}\right)$$

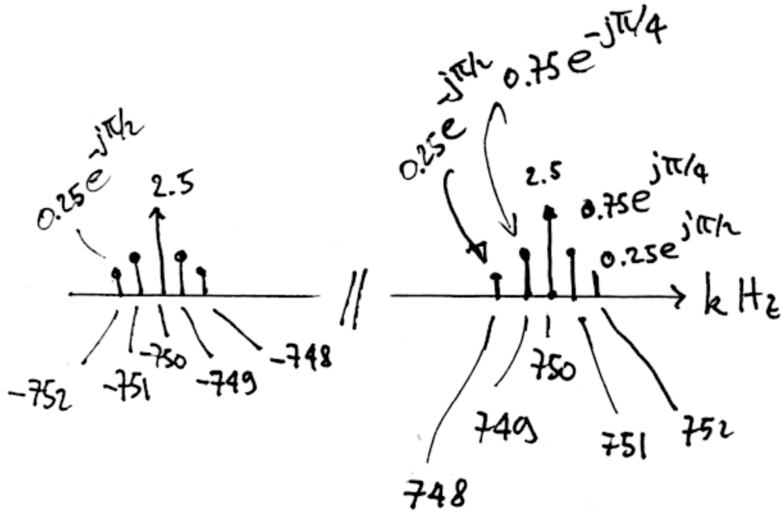
$$+ \frac{3}{2}\cos\left(2\pi \cdot 750 \cdot 10^3 t - 2000\pi t - \frac{\pi}{4}\right)$$

$$+ \frac{1}{2}\cos\left(2\pi \cdot 750 \cdot 10^3 t + 4000\pi t + \frac{\pi}{2}\right)$$

$$+ \frac{1}{2}\cos\left(2\pi \cdot 750 \cdot 10^3 t - 4000\pi t - \frac{\pi}{2}\right)$$

Spectral components :

frequency (rad/sec)	complex ampl.
$2\pi \cdot 750 \cdot 10^3$	$\frac{A}{2} = 2.5$
$-2\pi \cdot 750 \cdot 10^3$	$A/2 = 2.5$
$2\pi \cdot 751 \cdot 10^3$	$\frac{3}{4}e^{j\pi/4}$
$-2\pi \cdot 751 \cdot 10^3$	$\frac{3}{4}e^{-j\pi/4}$
$2\pi \cdot 749 \cdot 10^3$	$\frac{3}{4}e^{-j\pi/4}$
$-2\pi \cdot 749 \cdot 10^3$	$\frac{3}{4}e^{j\pi/4}$
$2\pi \cdot 752 \cdot 10^3$	$\frac{1}{4}e^{j\pi/2}$
$-2\pi \cdot 752 \cdot 10^3$	$\frac{1}{4}e^{-j\pi/2}$
$2\pi \cdot 748 \cdot 10^3$	$\frac{1}{4}e^{-j\pi/2}$
$-2\pi \cdot 748 \cdot 10^3$	$\frac{1}{4}e^{j\pi/2}$



$\underline{2.6}$ $$z(t) = Z e^{j\pi t/2}, \quad Z = e^{-j\pi/3}$$

a) $$\frac{d}{dt} z(t) = Z \frac{d}{dt} e^{j\pi t/2}$$

$$= Z \, j\pi/2 \, e^{j\pi t/2}$$

$$= \underbrace{Z \frac{\pi}{2} e^{j\frac{\pi}{2}}}_{= Q} e^{j\pi t/2}$$

$$\therefore \begin{cases} |Q| = |Z| \frac{\pi}{2} \\ \arg Q = \arg Z + \frac{\pi}{2} \end{cases}$$

$$Q = Z \frac{\pi}{2} e^{j\pi/2} = j \frac{\pi}{2} Z$$

$$\therefore |Q| = |e^{-j\pi/3}| \cdot \frac{\pi}{2} = \frac{\pi}{2}$$

b) $$\int_0^1 z(t)\,dt = \int_0^1 Z e^{j\pi t/2} dt = Z \int_0^1 e^{j\pi t/2} dt = \frac{2Z}{j\pi} e^{j\pi t/2} \Big|_0^1$$

$$= \frac{2Z}{j\pi} \left(e^{j\pi/2} - 1\right)' = \frac{2Z}{j\pi}(j-1) = \frac{2Z}{\pi} e^{-j\frac{\pi}{2}} . \sqrt{2}\, e^{j\,3\pi/4}$$

$$= \frac{2\sqrt{2}}{\pi} e^{j\pi/4} Z = \frac{2\sqrt{2}}{\pi} e^{j\pi/4} . e^{-j\pi/3} = \frac{2\sqrt{2}}{\pi} e^{-j\pi/12}$$

c) $$\int_{-1}^{1} |z^2(t)|\,dt = \int_{-1}^{1} \left|e^{j(\pi\frac{t}{2} - \pi/3)}\right|^2 dt = \int_{-1}^{1} dt = 2$$