Problem Set #12

ECE 2025

Problem 12.1* – Properties of impulses:

(a) $\left[e^{-t}\cos 2\pi t\right]\left[\delta(t)+2\delta(t-3)\right] = ?$

Here we use the property $f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$

So the answer is

$$e^{0}\cos(2\pi\cdot 0)\,\delta(t)+2e^{-3}\cos(6\pi)\,\delta(t-3)$$

$$= \delta(t)+2e^{-3}\delta(t-3)$$

(b) $\left[\delta(t)-u(t)\right]*\left[\delta(t-1)+\delta(t-2)\right] = ?$

Here we use the property $f(t)*\delta(t-t_1) = f(t-t_1)$

So the answer is

$$\delta(t-1)-u(t-1)+\delta(t-2)-u(t-2)$$

(c) $\displaystyle\int_{-\infty}^{\infty}\sin(2\pi\tau+\pi/3)\,\delta(t-\tau)\,d\tau = \sin(2\pi t+\pi/3)*\delta(t)$

$$= \sin(2\pi t+\pi/3)$$

(d) $\dfrac{d}{dt}\left[e^{-(t-1)}u(t-1)\right] = -e^{-(t-1)}u(t-1)+e^{-(t-1)}\delta(t-1)$

$$= \delta(t)-e^{-(t-1)}u(t-1)$$

We used two properties here $\dfrac{du}{dt} = \delta(t)$ \& $f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$

\+ the product rule for differentiation.

Problem 12.2* $\quad h(t) = \delta(t) - 10e^{-10t}u(t)$

(a) $H(j\omega)$ is the Fourier transform of $h(t)$ so,

$$H(j\omega) = 1 - \frac{10}{10+j\omega} = \frac{j\omega}{10+j\omega}$$

Here we used the FT pairs

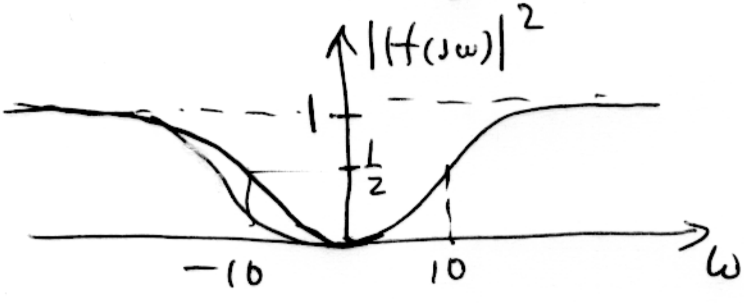
$$\delta(t) \Longleftrightarrow 1$$

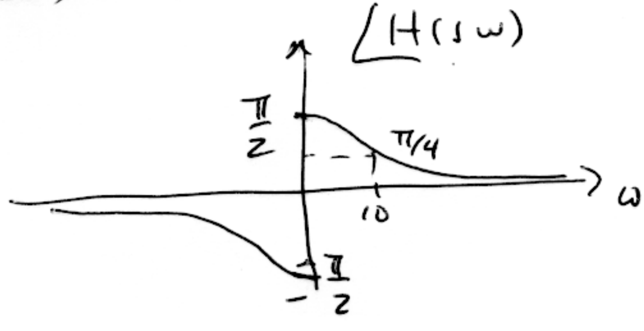
$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a+j\omega}$$

(b) $$|H(j\omega)|^2 = H(j\omega)H^*(j\omega) = \left(\frac{j\omega}{10+j\omega}\right)\left(\frac{-j\omega}{10-j\omega}\right) = \frac{\omega^2}{100+\omega^2}$$

$$\angle H(j\omega) = \angle j\omega - \angle a+j\omega$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{a}\right)$$





Problem 12.2 (c)

$x(t) = 100 + 20\cos(10t) + \delta(t-3)$

Here the key is to work the problem with the right tools.

$$100 \longmapsto 100\,H(0) = 0$$

$$20\cos(10t) = 10e^{j10t} + 10e^{-j10t} \longmapsto 10\,H(j10)e^{j10t} + 10\,H(-j10)e^{-j10t}$$

$$= \frac{j100}{10+j10}e^{j10t} + \frac{-j100}{10-j10}e^{-j10t}$$

$$= \frac{10e^{j\frac{\pi}{2}}}{\sqrt{2}e^{j\pi/4}}e^{j10t} + \frac{10e^{-j\pi/2}}{\sqrt{2}e^{-j\pi/4}}e^{-j10t}$$

$$= \frac{20}{\sqrt{2}}\cos(10t + \pi/4)$$

Finally

$$\delta(t-3) \longmapsto h(t-3) = \delta(t-3) - 10e^{-10(t-3)}u(t-3)$$

So by superposition,

$$y(t) = 0 + \frac{20}{\sqrt{2}}\cos(10t + \pi/4) + \delta(t-3) - 10e^{-10(t-3)}u(t-3)$$

Problem 12.3*

$$y(t) = \frac{1}{2}x(t) + x(t-4) + \frac{1}{2}x(t-8)$$

(a) $h(t)$ is what we get when $x(t) = \delta(t)$, so

$$h(t) = \frac{1}{2}\delta(t) + \delta(t-4) + \frac{1}{2}\delta(t-8)$$

(b) $$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2}\delta(t) + \delta(t-4) + \frac{1}{2}\delta(t-8)\right] e^{-j\omega t}\,dt$$

Using the properties of impulses,

$$H(j\omega) = \frac{1}{2} + e^{-j\omega 4} + \frac{1}{2}e^{-j\omega 8}$$

(c) Now we also know that $H(j\omega)$ is defined by the output when $x(t) = e^{j\omega t}$

so let $x(t) = e^{j\omega t}$

$$y(t) = \frac{1}{2}e^{j\omega t} + e^{j\omega(t-4)} + \frac{1}{2}e^{j\omega(t-8)}$$

$$= \underbrace{\left(\frac{1}{2} + e^{-j\omega 4} + \frac{1}{2}e^{-j\omega 8}\right)}_{H(j\omega)} e^{j\omega t}$$

which is what we got before.

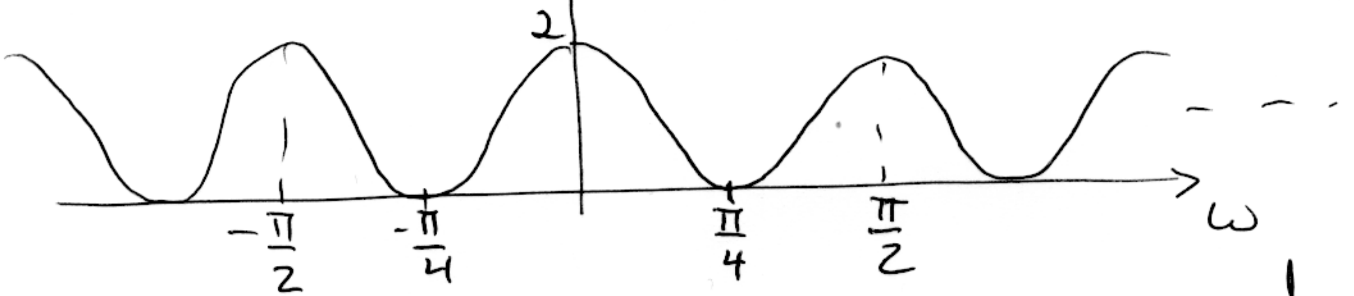
Problem 12.3 (c)

$$H(j\omega) = \frac{1}{2} + e^{-j\omega 4} + \frac{1}{2}e^{-j\omega 8}$$

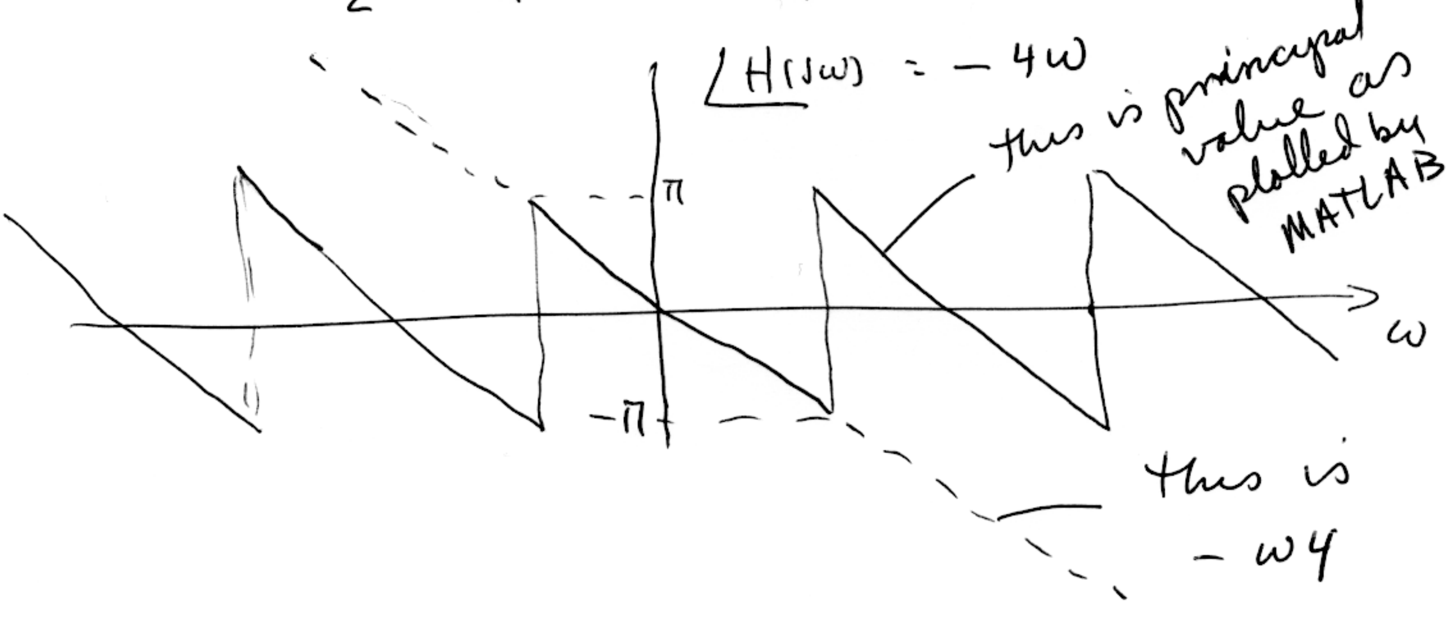
$$= e^{-j\omega 4}\left(\frac{1}{2}e^{j\omega 4} + 1 + \frac{1}{2}e^{-j4\omega}\right)$$

$$= \left[1 + \cos(\omega 4)\right]e^{-j\omega 4}$$

$$|H(j\omega)| = 1 + \cos(4\omega)$$



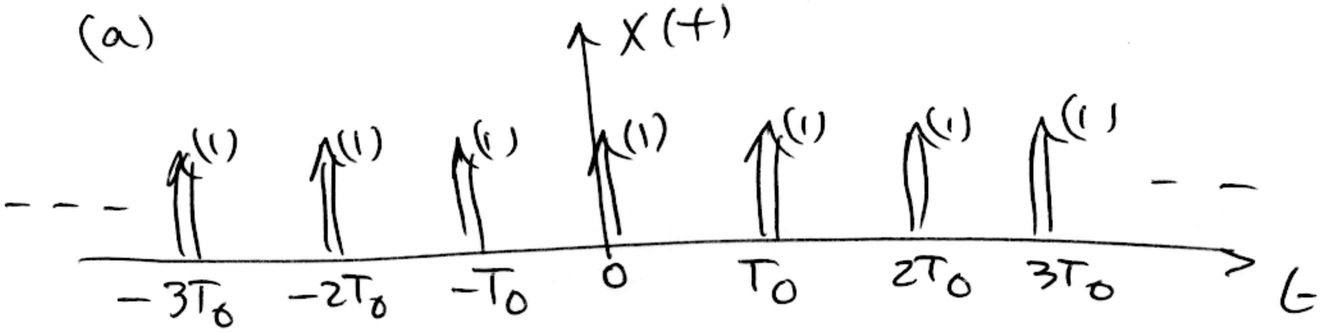
$$\angle H(j\omega) = -4\omega$$



12.4*

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

(a)



(b) $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{10}$ if $T_0 = 10$

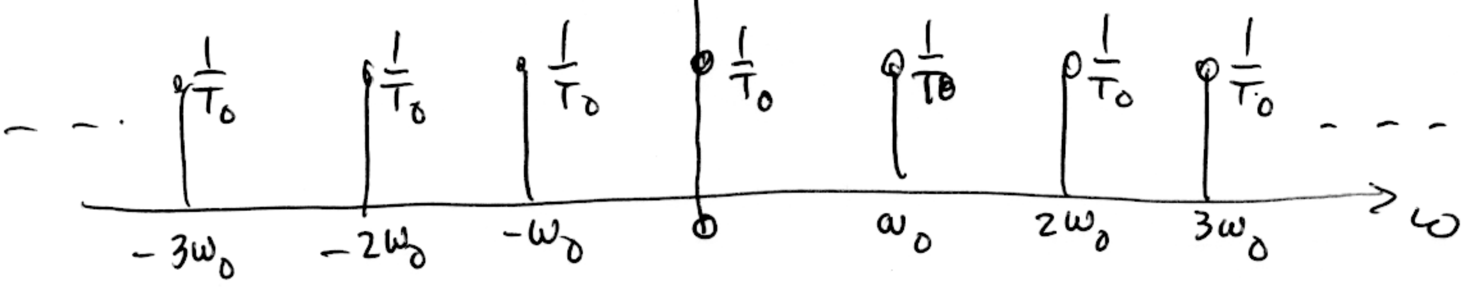
(c)

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-jk\omega_0 t}\,dt$$

$x(t) = \delta(t)$ for $-\frac{T_0}{2} < t < T_0/2$

$a_k = \frac{1}{T_0}$ for all k.

(d) Spectrum of $x(t)$



Problem 12.4(e)

(e) $H(jw) = \begin{cases} e^{-jw4} & |w| < w_{co} \\ 0 & |w| > w_{co} \end{cases}$

This is an ideal lowpass filter. The factor $e^{-jw4} \Rightarrow$ delay by 4.

Now if $w_{co} = \frac{\pi}{T_0}$, only the d.c. component passes through since.

The output signal is

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jkw_0) e^{jkw_0 t}$$

So $y(t) = \frac{1}{T_0} = \frac{1}{10} \quad -\infty < t < \infty$

if $w_{co} = \frac{\pi}{T_0}$

(f) If $w_{co} = 3\pi/T_0$ the cutoff is between w_0 and $2w_0$ so we get

$$y(t) = \frac{1}{T_0} + \frac{1}{T_0} e^{-jw_0 4} e^{jw_0 t} + \frac{1}{T_0} e^{jw_0 4} e^{-jw_0 t}$$

$$= \frac{1}{T_0}\left(1 + 2\cos(w_0(t-4))\right)$$

Prob 12.5

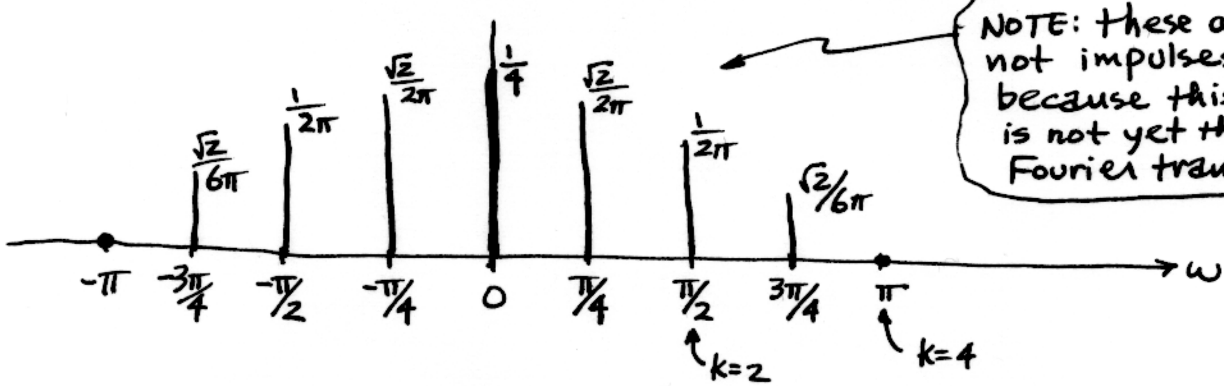
(a) $\omega_0 = 2\pi/T_0 = 2\pi/8 = \pi/4$ rad/sec

$$a_k = \frac{1}{8}\int_{\cancel{-4}\,-1}^{\cancel{4}\,1} 1\, e^{-jk\omega_0 t} = \frac{1}{8}\;\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\Bigg|_{-1}^{1}$$

$$= \frac{e^{-jk\omega_0} - e^{+jk\omega_0}}{-jk(2\pi)} = \frac{\sin k\omega_0}{k\pi} = \frac{\sin(k\pi/4)}{k\pi}$$

(b) Need to eval for $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$

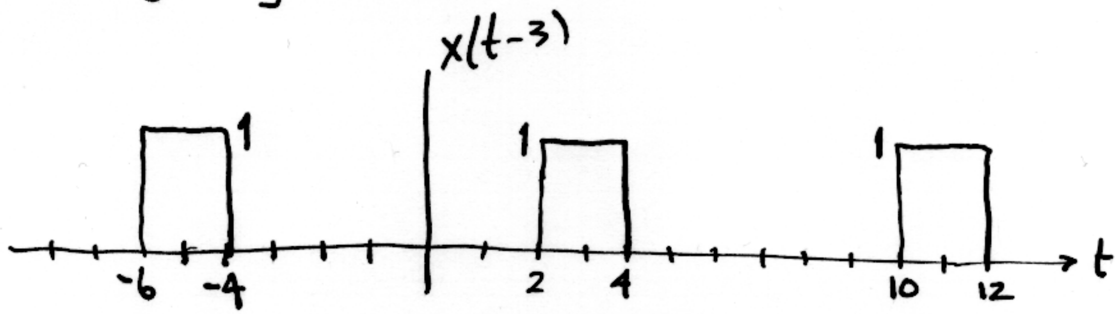
k	-4	-3	-2	-1	0	1	2	3	4
a_k	0	$\frac{\sqrt{2}}{6\pi}$	$\frac{1}{2\pi}$	$\frac{\sqrt{2}}{2\pi}$	$1/4$	$\frac{\sqrt{2}}{2\pi}$	$\frac{1}{2\pi}$	$\frac{\sqrt{2}}{6\pi}$	0



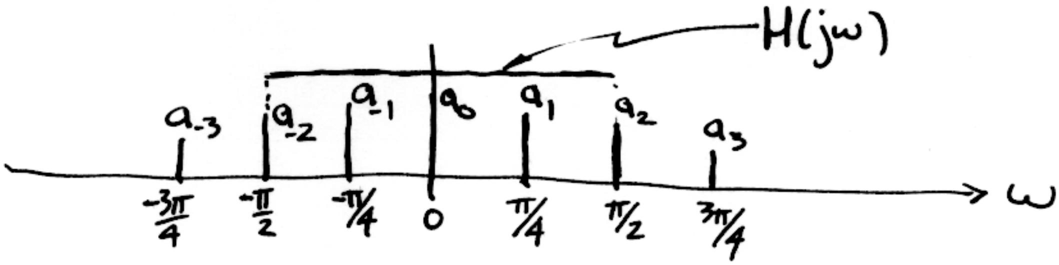
NOTE: these are not impulses because this is not yet the Fourier transform

(c) $e^{-j\omega}$ is frequency response when $h(t) = \delta(t-3)$

$\Rightarrow$ delay by 3 secs.



(d)



Since $H(j\omega)$ is zero for $|\omega| > \pi/2$, the spectrum components at $\omega = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}$ & $\pi/2$ are PASSED. The others are REJECTED.

NOTE: the bandedge of the filter is at $\omega = \pm\pi/2$ so it is hard to tell if the a_2 & a_2 components should be included. Either answer is OK

$$y(t) = \frac{1}{4} + \underbrace{\frac{\sqrt{2}}{2\pi} e^{j\frac{\pi}{4}t} + \frac{\sqrt{2}}{2\pi} e^{-j\frac{\pi}{4}t}}_{=\frac{\sqrt{2}}{\pi}\cos(\frac{\pi}{4}t)} + \underbrace{\frac{1}{2\pi} e^{j\frac{\pi}{2}t} + \frac{1}{2\pi} e^{-j\frac{\pi}{2}t}}_{\frac{1}{\pi}\cos(\frac{\pi}{2}t)}$$

If you include the components at $\omega = \pm\pi/2$ then

$$y(t) = \frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\left(\frac{\pi}{4}t\right) + \frac{1}{\pi}\cos\left(\frac{\pi}{2}t\right)$$

If not, then

$$y(t) = \frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\left(\frac{\pi}{4}t\right)$$

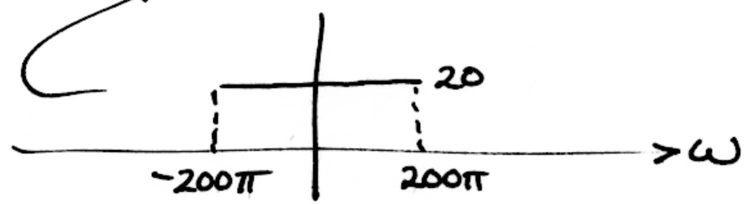
12.6

(a) $x(t) = \delta(t-5)$

$$\text{F.T.}\{\delta(t)\} = 1$$

$$\text{F.T.}\{\delta(t-5)\} = e^{-j\omega 5} \cdot \text{F.T.}\{\delta(t)\} = e^{-j5\omega}$$

(b) $$\text{F.T.}\left\{20\,\frac{\sin(200\pi t)}{\pi t}\right\} = \begin{cases} 20 & \text{if } |\omega| < 200\pi \\ 0 & \text{elsewhere} \end{cases}$$



$$X(j\omega) = e^{-j10\omega} \cdot \text{FT}\left\{20\,\frac{\sin(200\pi t)}{\pi t}\right\} = \begin{cases} 20e^{-j10\omega}, & |\omega| < 200\pi \\ 0 & \text{else} \end{cases}$$

(c) $$x(t) = e^{-4t}u(t) - e^{-40}\,e^{-4(t-10)}u(t-10)$$

$$\frac{1}{4+j\omega} - e^{-40}\,\frac{e^{-j10\omega}}{4+j\omega} = \frac{1 - e^{-10(4+j\omega)}}{4+j\omega}$$